Cognition and conditionals

Niki Pfeifer

in collaboration with
Leonhard Kratzer, Andy Fugard & Gernot D. Kleiter

Department of Psychology
University of Salzburg

www.users.sbg.ac.at/~pfeifern/
Outline

- Conditionals in psychology
  - Indicative conditionals
  - Uncertain conditionals
- Mental probability logic
  - Wason's selection task
  - Truth table task
  - Paradoxes of the material conditional
- Conclusions
Conditionals in psychology: Indicative conditionals

Three prominent psychological predictions of how people interpret “If $A$, then $B$”:

- Material conditional, $A \supset B$
- Conjunction, $A \land B$
- Conditional event, $B|A$
Conditionals in psychology: Indicative conditionals

Three prominent psychological predictions of how people interpret “If A, then B”:

- Material conditional, $A \supset B$
- Conjunction, $A \land B$
- Conditional event, $B\mid A$

<table>
<thead>
<tr>
<th></th>
<th>$A$</th>
<th>$B$</th>
<th>$A \supset B$</th>
<th>$A \land B$</th>
<th>$B\mid A$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s_1$</td>
<td>true</td>
<td>true</td>
<td>true</td>
<td>true</td>
<td>true</td>
</tr>
<tr>
<td>$s_2$</td>
<td>true</td>
<td>false</td>
<td>false</td>
<td>false</td>
<td>false</td>
</tr>
<tr>
<td>$s_3$</td>
<td>false</td>
<td>true</td>
<td>true</td>
<td>false</td>
<td>undetermined</td>
</tr>
<tr>
<td>$s_4$</td>
<td>false</td>
<td>false</td>
<td>true</td>
<td>false</td>
<td>undetermined</td>
</tr>
</tbody>
</table>
Conditionals in psychology: Indicative conditionals

Three prominent psychological predictions of how people interpret “If $A$, then $B$”:

- Material conditional, $A \supset B$
- Conjunction, $A \land B$
- Conditional event, $B|A$

|   | $A$  | $B$  | $A \supset B$ | $A \land B$ | $B|A$  |
|---|------|------|---------------|-------------|--------|
| $s_1$ | true | true | true          | true        | true   |
| $s_2$ | true | false| false         | false       | false  |
| $s_3$ | false| true | true          | false       | undetermined |
| $s_4$ | false| false| true          | false       | undetermined |
Conditionals in psychology: Uncertain conditionals

\[ P(A \supset B) \]
Conditionals in psychology: Uncertain conditionals

$P(A ⊃ B)$

$P(A \land B)$

Probabilistic extension
of the mental model theory

Johnson-Laird et al.
Conditionals in psychology: Uncertain conditionals

\[ P(A \supset B) \]
\[ P(A \land B) \]

Probabilistic extension of the mental model theory
Johnson-Laird et al.

Theoretical problems:
Paradoxes of the material conditional:
e.g., from \( B \) infer \( \text{if } A, \text{ then } B \)
Conditionals in psychology: Uncertain conditionals

\[ P(A \supset B) \]
\[ P(A \land B) \]

Theoretical problems:

Paradoxes of the material conditional:

e.g., from \( B \) infer \( \text{if } A, \text{ then } B \)

The material conditional is not a genuine conditional

\[ (A \supset B) \iff (\neg A \lor B) \]

Probabilistic extension of the mental model theory

Johnson-Laird et al.
Conditionals in psychology: Uncertain conditionals

Probabilistic extension
of the *mental model* theory

Johnson-Laird et al.
Conditionals in psychology: Uncertain conditionals

\[ P(A \supset B) \]
\[ P(A \land B) \]

Probabilistic extension of the *mental model* theory

Johnson-Laird et al.

\[ P(B \mid A) \]

Theoretical problems solved:

- No paradoxes of the material conditional:
  - From \( P(B) = x \) infer \( P(B \mid A) \in [0, 1] \)
Conditionals in psychology: Uncertain conditionals

\[ P(A \supset B) \quad P(A \land B) \quad P(B|A) \]

Theoretical problems \textbf{solved}:

- No paradoxes of the material conditional:
  - From \( P(B) = x \) infer \( P(B|A) \in [0, 1] \)
  - But: from \( P(B) = x \) infer \( P(A \supset B) \in [x, 1] \)

Probabilistic extension of the \textit{mental model} theory

Johnson-Laird et al.
Conditionals in psychology: Uncertain conditionals

$P(A \supset B)$

$P(A \land B)$

$P(B|A)$

Probabilistic extension of the mental model theory

Johnson-Laird et al.

Theoretical problems solved:

No paradoxes of the material conditional:

From $P(B) = x$ infer $P(B|A) \in [0, 1]$.

But: from $P(B) = x$ infer $P(A \supset B) \in [x, 1]$.

The conditional event $B|A$ is a genuine conditional!
Conditionals in psychology: Uncertain conditionals

\[ P(A \supset B) \]
\[ P(A \land B) \]
\[ P(B \mid A) \]

Probabilistic extension of the *mental model* theory
Johnson-Laird et al.

Empirical Result:
\[ P(B \mid A) \] best predictor for “if A, then B”
Evans, Over et al.
Oberauer et al.
Liu
Conditionals in psychology: Uncertain conditionals

\[ P(A \supset B) \]
\[ P(A \land B) \]
\[ P(B|A) \]

Probabilistic relation between premise(s) and conclusion
Chater, Oaksford et al.
Liu et al.

Probabilistic extension of the *mental model* theory
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Empirical Result:
\[ P(B|A) \] best predictor for “if A, then B”
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Conditionals in psychology: Uncertain conditionals

Probabilistic relation between premise(s) and conclusion
- Chater, Oaksford et al.
- Liu et al.

Probabilistic extension of the mental model theory
- Johnson-Laird et al.

Deductive relation between premise(s) and conclusion
- Mental probability logic
  - Pfeifer & Kleiter

Empirical Result:
- $P(B|A)$ best predictor for “if A, then B”
  - Evans, Over et al.
  - Oberauer et al.
  - Liu
Mental probability logic

- embedded in a probability logic framework
Mental probability logic

- embedded in a probability logic framework
- the indicative “if $A$, then $B$” is interpreted as a nonmonotonic conditional:
Mental probability logic

- embedded in a probability logic framework
- the indicative “if A, then B” is interpreted as a nonmonotonic conditional:
  - A, normally B iff \( P(B|A) = \text{high} \)
Mental probability logic

- embedded in a probability logic framework
- the indicative “if A, then B” is interpreted as a nonmonotonic conditional:
  - A, normally B iff \( P(B|A) = \text{high} \)
- competence theory
Mental probability logic

- embedded in a probability logic framework
- the indicative "if $A$, then $B$" is interpreted as a nonmonotonic conditional:
  - $A$, normally $B$ iff $P(B|A) = \text{high}$
- competence theory
- the uncertainty of the conclusion is inferred deductively from the uncertainty of the premises, e.g.: 
Mental probability logic

- embedded in a probability logic framework
- the indicative “if A, then B” is interpreted as a nonmonotonic conditional:
  - A, normally B iff $P(B|A) = \text{high}$
- competence theory
- the uncertainty of the conclusion is inferred deductively from the uncertainty of the premises, e.g.:
  - Premises:
    - $P(B|A) = x$, $P(A) = y$  $\models$  Conclusion:
    - $P(B) \in [xy, xy + 1 - y]$
Mental probability logic

- embedded in a probability logic framework
- the indicative “if $A$, then $B$” is interpreted as a nonmonotonic conditional:
  - $A$, normally $B$  iff  $P(B|A) = high$
- competence theory
- the uncertainty of the conclusion is inferred deductively from the uncertainty of the premises, e.g.:

\[
\begin{align*}
\text{premises} & : & P(B|A) = x, & P(A) = y & \implies & P(B) \in [xy, xy + 1 - y] \\
\text{conclusion} & :
\end{align*}
\]
- premises are evaluated by point values, intervals or second order probability distributions
Mental probability logic

- embedded in a probability logic framework
- the indicative “*if A, then B*” is interpreted as a nonmonotonic conditional:
  - $A$, normally $B$ iff $P(B|A) = \text{high}$
- competence theory
- the uncertainty of the conclusion is inferred deductively from the uncertainty of the premises, e.g.:
  \[
  \begin{align*}
  &\text{premises} \\
  \quad P(B|A) = x, \quad P(A) = y &\text{conclusion} \\
  \quad P(B) \in [xy, xy + 1 - y] \\
  \end{align*}
  \]
  - premises are evaluated by point values, intervals or second order probability distributions
- coherence
Coherence

- de Finetti, and \{Lad, Walley, Scozzafava, Coletti, Gilio,\ldots\}
- degrees of belief
- complete algebra is not required
- conditional probability, $P(B|A)$, is primitive
- zero probabilities are exploited to reduce the complexity
- imprecision
- provides semantics for System P
Wasons selection task
Wasons selection task

If there is a vowel on the one side, then there is an even number on the other side.
Wason's selection task

If there is a vowel on the one side, then there is an even number on the other side.
Wasons selection task

If there is a vowel on the one side ($A$), then there is an even number on the other side ($B$).

\[ \text{E} \quad \text{K} \quad 4 \quad 7 \]

\[ A \quad \neg A \quad B \quad \neg B \]
Wason's selection task

If there is a vowel on the one side \((A)\), then there is an even number on the other side \((B)\).

\[ \begin{array}{c}
E & K & 4 & 7 \\
A & \neg A & B & \neg B \\
\end{array} \]

- 46% choose both, the \(A\)- and the \(B\)-card \((\text{Wason & Johnson-Laird, 1972})\)
Wason's selection task

If there is a vowel on the one side \((A)\), then there is an even number on the other side \((B)\).

- 46% choose both, the \(A\)- and the \(B\)-card (Wason & Johnson-Laird, 1972)
- 33% choose the \(A\)-card (Wason & Johnson-Laird, 1972)
Wason's selection task

If there is a vowel on the one side ($A$), then there is an even number on the other side ($B$).

- 46% choose both, the $A$- and the $B$-card (Wason & Johnson-Laird, 1972)
- 33% choose the $A$-card (Wason & Johnson-Laird, 1972)
- 4% choose both, the $A$- and the $\neg B$-card (Wason & Johnson-Laird, 1972)
Wason's selection task

If there is a vowel on the one side ($A$), then there is an even number on the other side ($B$).

\[
\begin{array}{c|c|c|c}
A & \neg A & B & \neg B \\
\hline
A \supset B & A \supset B & A \supset B & A \supset B \\
B & \neg B & A & \neg A \\
\checkmark & (MP) & (DA) & (AC) & (MT)
\end{array}
\]

- 46% choose both, the $A$- and the $B$-card (Wason & Johnson-Laird, 1972)
- 33% choose the $A$-card (Wason & Johnson-Laird, 1972)
- 4% choose both, the $A$- and the $\neg B$-card (Wason & Johnson-Laird, 1972)
Wason's selection task

If there is a vowel on the one side ($A$), then there is an even number on the other side ($B$).

\[
\begin{array}{cccc}
A & \neg A & B & \neg B \\
A \equiv B & A \equiv B & A \equiv B & A \equiv B \\
B & \neg B & A & \neg A \\
\checkmark & \checkmark & \checkmark & \checkmark \\
\end{array}
\]

- 46% choose both, the $A$- and the $B$-card (Wason & Johnson-Laird, 1972)
- 33% choose the $A$-card (Wason & Johnson-Laird, 1972)
Wason's selection task

If there is a vowel on the one side (A), then there is an even number on the other side (B).

46% choose both, the A- and the B-card (Wason & Johnson-Laird, 1972)

33% choose the A-card (Wason & Johnson-Laird, 1972)
Wason's selection task

If there is a vowel on the one side ($A$), then there is an even number on the other side ($B$).

<table>
<thead>
<tr>
<th></th>
<th>E</th>
<th>K</th>
<th>4</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>$\neg A$</td>
<td>$B$</td>
<td>$\neg B$</td>
<td></td>
</tr>
<tr>
<td>$A \land B$</td>
<td>$A \land B$</td>
<td>$A \land B$</td>
<td>$A \land B$</td>
<td></td>
</tr>
<tr>
<td>$B$</td>
<td>$\neg B$</td>
<td>$A$</td>
<td>$\neg A$</td>
<td></td>
</tr>
<tr>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td></td>
</tr>
</tbody>
</table>

- 46% choose both, the $A$- and the $B$-card (Wason & Johnson-Laird, 1972)
- 33% choose the $A$-card (Wason & Johnson-Laird, 1972)
Wason's selection task

If there is a vowel on the one side \((A)\),
then there is an even number on the other side \((B)\).

\[
\begin{align*}
P(A) &= 1 \\
P(\neg A) &= 1 \\
P(B) &= 1 \\
P(\neg B) &= 1 \\
P(A \supset B) &= 1 \\
P(\neg A \supset B) &= 1 \\
P(\neg B \supset A) &= 1 \\
P(\neg A \supset \neg B) &= 1 \\
P(B) &= 1 \\
P(\neg B) &\in [0, 1] \\
P(A) &\in [0, 1] \\
P(\neg A) &= 1
\end{align*}
\]
Wason's selection task

If there is a vowel on the one side ($A$), then there is an even number on the other side ($B$).

<p>| | | | |</p>
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<tr>
<td>E</td>
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<td>7</td>
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- $P(A) = 1$
- $P(\neg A) = 1$
- $P(B) = 1$
- $P(\neg B) = 1$
- $P(B|A) = 1$
- $P(B|A) = 1$
- $P(B|A) = 1$
- $P(B|A) = 1$
- $P(B) = 1$
- $P(\neg B) \in [0, 1]$
- $P(A) \in [0, 1]$
- $P(\neg A) = 1$

✓ ✓
Wason's selection task

If there is a vowel on the one side \((A)\), then there is an even number on the other side \((B)\).

\[
\begin{align*}
P(A) &= 1 & P(\neg A) &= 1 & P(B) &= 1 & P(\neg B) &= 1 \\
P(A \land B) &= 1 & P(A \land B) &= 1 & P(A \land B) &= 1 & P(A \land B) &= 1 \\
\end{align*}
\]

\(P(B) = 1\) \(\checkmark\) \(\text{incoherent!}\) \(P(A) = 1\) \(\checkmark\) \(\text{incoherent!}\)
Truth table task
Task AA, SP condition

If there is a circle on the screen, then the circle is black.

Does the shape on the screen speak for the assertion in the box?

☐ speaks against  ☐ neither/nor  ☐ speaks for
Task AA, PS condition

If there is a black shape on the screen, then it is a circle.

Does the shape on the screen speak for the assertion in the box?

☐ speaks against ☐ neither/nor ☐ speaks for
Task AN, SP condition

If there is a circle on the screen, then the circle is black.

Does the shape on the screen speak for the assertion in the box?

☐ speaks against  ☐ neither/nor  ☐ speaks for
Task NA, SP condition

If there is a circle on the screen, then the circle is black.

Does the shape on the screen speak for the assertion in the box?

☐ speaks against
☐ neither/nor
☐ speaks for
**Task NN, SP condition**

*If there is a circle on the screen, then the circle is black.*

Does the shape on the screen speak for the assertion in the box?

- □ speaks against
- □ neither/nor
- □ speaks for
Design

- Two conditions: SP ($n_1 = 18$) and PS ($n_2 = 18$)
- 16 target tasks: 4 conditionals × 4 truth table cases
- Order of tasks:

<table>
<thead>
<tr>
<th>Conditional in box</th>
<th>Shape on screen</th>
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<tbody>
<tr>
<td>If circle, then black</td>
<td>●</td>
<td>target AA</td>
</tr>
<tr>
<td>If circle, then black</td>
<td>○</td>
<td>target AN</td>
</tr>
<tr>
<td>If circle, then black</td>
<td>▲</td>
<td>target NA</td>
</tr>
<tr>
<td>If circle, then black</td>
<td>△</td>
<td>target NN</td>
</tr>
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Design

- **Two conditions**: SP \((n_1 = 18)\) and PS \((n_2 = 18)\)
- **16 target tasks**: 4 conditionals \(\times\) 4 truth table cases
- **Order of tasks:**

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<td>▲</td>
<td>target NA</td>
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<tr>
<td>If circle, then black</td>
<td>△</td>
<td>target NN</td>
</tr>
<tr>
<td>counterfactual</td>
<td>broken screen</td>
<td>filler item</td>
</tr>
</tbody>
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Design

- Two conditions: SP ($n_1 = 18$) and PS ($n_2 = 18$)
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<td>If circle, then white</td>
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<td>target NN</td>
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<tr>
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<tr>
<td>If triangle, then white</td>
<td>△</td>
<td>target AA</td>
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## Results: Mean Response Percentages

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<thead>
<tr>
<th>Group</th>
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<tr>
<td></td>
<td>AA</td>
<td>AN</td>
</tr>
<tr>
<td>SP</td>
<td>speaks against</td>
<td>2.78</td>
</tr>
<tr>
<td></td>
<td>neither/nor</td>
<td>4.17</td>
</tr>
<tr>
<td></td>
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<td>93.06</td>
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Representation as a conditional event (\(\cdot|\cdot\))
Results: Mean Response Percentages

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<tr>
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<td>0.00</td>
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<tr>
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<td>neither/nor</td>
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<tr>
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Representation as a conditional event (·|·)
## Results: Mean Response Percentages

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<tr>
<td></td>
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Representation as a conjunction \((\cdot \land \cdot)\)
## Results: Mean Response Percentages

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<td>4.17</td>
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<td>93.06</td>
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<tr>
<td>PS</td>
<td>speaks against</td>
<td>0.00</td>
</tr>
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<td></td>
<td>neither/nor</td>
<td>5.56</td>
</tr>
<tr>
<td></td>
<td>speaks for</td>
<td>94.44</td>
</tr>
</tbody>
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Representation as a material conditional ($\cdot \supset \cdot$)
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<td>AA</td>
<td>AN</td>
</tr>
<tr>
<td>SP</td>
<td>speaks against 2.78</td>
<td>86.11</td>
</tr>
<tr>
<td></td>
<td>neither/nor 4.17</td>
<td>11.11</td>
</tr>
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<td></td>
<td>speaks for 93.06</td>
<td>2.78</td>
</tr>
<tr>
<td>PS</td>
<td>speaks against 0.00</td>
<td>91.67</td>
</tr>
<tr>
<td></td>
<td>neither/nor 5.56</td>
<td>6.94</td>
</tr>
<tr>
<td></td>
<td>speaks for 94.44</td>
<td>1.39</td>
</tr>
</tbody>
</table>

Not yet clear what’s going on here.
Paradoxes of the material conditional
Two paradoxes of the material conditional (conditional introduction): 

“If $A$, then $B$” interpreted as “$A \supset B$”

\[ B \vdash A \supset B \]

\[\begin{array}{c}
\text{log. valid} \\
A \supset B
\end{array}\]
Two paradoxes of the material conditional (conditional introduction):

“If A, then B” interpreted as “A ⊃ B”

Not: The moon is made of green cheese

If the moon is made of green cheese, then 2 + 2 = 4
Two paradoxes of the material conditional (conditional introduction):

“If A, then B” interpreted as “A ⊃ B”

¬: The moon is made of green cheese

If the moon is made of green cheese, then 2 + 2 = 4

Mental model theory postulates that subjects represent “basic conditionals” “If A, then B” as

- implicit mental models:

```
A   B
...
```

…truth conditions of the conjunction, A ∧ B
Two paradoxes of the material conditional (conditional introduction):

“\textbf{If A, then B}” interpreted as “\(A \supset B\)”

\begin{itemize}
  \item \(\not\) \textbf{Not}: The moon is made of green cheese
  \begin{itemize}
    \item \textbf{log. valid}
  \end{itemize}
  \item \(\therefore\) \textbf{If} the moon is made of green cheese, \textbf{then} \(2 + 2 = 4\)
\end{itemize}

Mental model theory postulates that subjects represent “\textbf{basic conditionals}” “\textbf{If A, then B}” as

\begin{itemize}
  \item implicit mental models
  \item explicit mental models:
    \begin{itemize}
      \item \(A\) \hspace{1cm} \(B\)
      \item \(\text{not-}A\) \hspace{1cm} \(B\)
      \item \(\text{not-}A\) \hspace{1cm} \(\text{not-}B\)
    \end{itemize}
\end{itemize}

\(\ldots\) truth conditions of the \textbf{material conditional}, \(A \supset B\)
Two paradoxes of the material conditional, $A \supset B$

Example (Paradox 1)

$B$ : If $A$, then $B$

<table>
<thead>
<tr>
<th>Premise</th>
<th>Conclusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B$</td>
<td>$A \supset B$</td>
</tr>
<tr>
<td>$P(B) = x$</td>
<td>$P(A \supset B) \in [x, 1]$</td>
</tr>
<tr>
<td>$P(B) = x$</td>
<td>$P(A \land B) \in [0, x]$</td>
</tr>
<tr>
<td>$P(B) = x$</td>
<td>$P(B</td>
</tr>
</tbody>
</table>
Two paradoxes of the material conditional, $A \supset B$

Example (Paradox 1)

$B \therefore$ If $A$, then $B$
$B \therefore$ If $A$, then not-$B$

<table>
<thead>
<tr>
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<tbody>
<tr>
<td>$B$</td>
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<td>$P(A \land B) \in [0, x]$</td>
</tr>
<tr>
<td>$P(B) = x$</td>
<td>$P(B\mid A) \in [0, 1]$</td>
</tr>
</tbody>
</table>

| $B$     | $A \supset \neg B$ |
| $P(B) = x$ | $P(A \supset \neg B) \in [1-x, 1]$ |
| $P(B) = x$ | $P(A \land \neg B) \in [0, 1-x]$ |
| $P(B) = x$ | $P(\neg B\mid A) \in [0, 1]$ |
Two paradoxes of the material conditional, $A \supset B$

Example (Paradox 1)

$B :. \text{If } A, \text{ then } B$

$B :. \text{If } A, \text{ then not-}B$

<table>
<thead>
<tr>
<th>Premise</th>
<th>Conclusion</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B$</td>
<td>$A \supset B$</td>
<td>(logically valid)</td>
</tr>
<tr>
<td>$P(B) = 1$</td>
<td>$P(A \supset B) = 1$</td>
<td>(prob. informative)</td>
</tr>
<tr>
<td>$P(B) = 1$</td>
<td>$P(A \land B) \in [0, 1]$</td>
<td>(pract. non-informative)</td>
</tr>
<tr>
<td>$P(B) = 1$</td>
<td>$P(B</td>
<td>A) \in [0, 1]$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Premise</th>
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<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B$</td>
<td>$A \supset \lnot B$</td>
<td>(not logically valid)</td>
</tr>
<tr>
<td>$P(B) = 1$</td>
<td>$P(A \supset \lnot B) \in [0, 1]$</td>
<td>(pract. non-informative)</td>
</tr>
<tr>
<td>$P(B) = 1$</td>
<td>$P(A \land \lnot B) = 0$</td>
<td>(prob. informative)</td>
</tr>
<tr>
<td>$P(B) = 1$</td>
<td>$P(\lnot B</td>
<td>A) \in [0, 1]$</td>
</tr>
</tbody>
</table>
Two paradoxes of the material conditional, $A \supset B$

Example (Paradox 2)

*Not-*$A : \text{ If } A, \text{ then } B$

<table>
<thead>
<tr>
<th>Premise</th>
<th>Conclusion</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$\neg A$</td>
<td>$A \supset B$</td>
<td>(logically valid)</td>
</tr>
<tr>
<td>$P(\neg A) = x$</td>
<td>$P(A \supset B) \in [x, 1]$</td>
<td>(prob. informative)</td>
</tr>
<tr>
<td>$P(\neg A) = x$</td>
<td>$P(A \land B) \in [0, 1 - x]$</td>
<td>(prob. informative)</td>
</tr>
<tr>
<td>$P(\neg A) = x$</td>
<td>$P(B</td>
<td>A) \in [0, 1]$</td>
</tr>
</tbody>
</table>
Two paradoxes of the material conditional, \( A \supset B \)

**Example (Paradox 2)**

\[
\text{Not-}A : \text{ If } A, \text{ then } B \\
\text{Not-}A : \text{ If } A, \text{ then not-}B
\]

<table>
<thead>
<tr>
<th><strong>Premise</strong></th>
<th><strong>Conclusion</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>( \neg A )</td>
<td>( A \supset B ) (logically valid)</td>
</tr>
<tr>
<td>( P(\neg A) = x )</td>
<td>( P(A \supset B) \in [x, 1] ) (prob. informative)</td>
</tr>
<tr>
<td>( P(\neg A) = x )</td>
<td>( P(A \land B) \in [0, 1 - x] ) (prob. informative)</td>
</tr>
<tr>
<td>( P(\neg A) = x )</td>
<td>( P(B</td>
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<td>( P(A \supset \neg B) \in [x, 1] ) (prob. informative)</td>
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<td>( P(A \land \neg B) \in [0, 1 - x] ) (prob. informative)</td>
</tr>
<tr>
<td>( P(\neg A) = x )</td>
<td>( P(\neg B</td>
</tr>
</tbody>
</table>
Experimental results
Paradox 1:  $B \therefore \text{If } A, \text{ then } B$

Simon works in a factory that produces playing cards. He is responsible for what is printed on the cards. On each card, there is a shape (triangle, square, . . .) of a certain color (green, blue, . . .), like:

- green triangle, green square, green circle, . . .
- blue triangle, blue square, . . .
- red triangle, . . .
Paradox 1:  \( B \therefore A \)  If \( A \), then \( B \)

Simon works in a factory that produces playing cards. He is responsible for what is printed on the cards. On each card, there is a shape (triangle, square, . . .) of a certain color (green, blue, . . .), like:

- green triangle, green square, green circle, . . .
- blue triangle, blue square, . . .
- red triangle, . . .

Imagine that a card got stuck in the printing machine. Simon cannot see what is printed on this card. Since Simon did observe the card production during the whole day, he is

\[ A \] Pretty sure: There is a square on this card.

Considering \[ A \], how certain can Simon be that the following sentence is true?

If there is a red shape on this card, then there is a square on this card.
Paradox 1: $B \therefore \text{If } A, \text{ then } B$

Pretty sure: There is a square on this card.

Considering $A$, how certain can Simon be that the following sentence is true?

If there is a red shape on this card, then there is a square on this card.

Considering $A$, can Simon infer—at all—how certain he can be, that the sentence in the box is true?

☐ NO, Simon cannot infer his certainty.
☐ YES, Simon can infer his certainty.
Paradox 1: \( B \therefore \text{ If } A, \text{ then } B \)

\[ A \] Pretty sure: There is a square on this card.

Considering \[ A \], how certain can Simon be that the following sentence is true?

If there is a red shape on this card, then there is a square on this card.

Considering \[ A \], can Simon infer—at all—how certain he can be, that the sentence in the box is true?

\( \square \) NO, Simon cannot infer his certainty.

\( \square \) YES, Simon can infer his certainty.

In case you ticked YES, please fill in

\( \square \) Simon can be pretty sure that the sentence in the box is false.

\( \square \) Simon can be pretty sure that the sentence in the box is true.
Paradox 1 \( (n_1 = 16) \)

Frequencies \((n=16)\):

- Non-informative
- Yes
- No

\[
\begin{align*}
\text{pretty sure} & & \text{absolutely certain} \\
\land & & \\
\loom & & \loom
\end{align*}
\]

Paradox 1

\[
\begin{align*}
&: B \quad \therefore A \rightarrow B
\end{align*}
\]
Paradox 1 ($n_3 = 19$)

Frequencies ($n=19$)

First Task | Second Task

Non-informative

Yes

No

\[ \Box \; \& \; : \; B \; \therefore \; A \rightarrow B \]
negated Paradox 1 ($n_3 = 19$)

Frequencies ($n=19$)

- Non-informative
- Yes
- No

$\neg B \vdash B \therefore A \rightarrow \neg B$

Legend:
- ■ First Task
- □ Second Task
Paradox 2 \( (n_2 = 15) \)

Frequencies \((n=15)\)

- Non-informative
- Yes
- No

- pretty sure
- absolutely certain

\[ \neg A \quad \therefore A \rightarrow B \]
Paradox 2 \((n_4 = 20)\)

<table>
<thead>
<tr>
<th></th>
<th>First Task</th>
<th>Second Task</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Non-informative</strong></td>
<td>17</td>
<td></td>
</tr>
<tr>
<td><strong>Yes</strong></td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td><strong>No</strong></td>
<td>10</td>
<td>0</td>
</tr>
</tbody>
</table>

![Bar chart showing frequencies for Paradox 2](image-url)
negated Paradox 2 ($n_2 = 15$)

Frequencies ($n=15$)

- Non-informative
- Yes
- No

\[
\begin{align*}
\neg A & \implies \neg B \\
\end{align*}
\]
negated Paradox 2 ($n_4 = 20$)

```
Frequencies (n=20)

Non-informative

First Task    Second Task

negated Paradox 2

\[ \neg A \quad \therefore A \rightarrow \neg B \]
```
Complement

If $A$, then $B$  \quad \therefore \quad$ If $A$, then $\neg B$

<table>
<thead>
<tr>
<th>Premise</th>
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<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$A \supset B$</td>
<td>$A \supset \neg B$</td>
<td>(not logically valid)</td>
<td></td>
</tr>
<tr>
<td>$P(A \supset B) = x$</td>
<td>$P(A \supset \neg B) \in [1 - x, 1]$</td>
<td>(prob. informative)</td>
<td></td>
</tr>
<tr>
<td>$P(A \land B) = x$</td>
<td>$P(A \land \neg B) \in [0, 1 - x]$</td>
<td>(prob. informative)</td>
<td></td>
</tr>
<tr>
<td>$P(B</td>
<td>A) = x$</td>
<td>$P(\neg B</td>
<td>A) = 1 - x$</td>
</tr>
</tbody>
</table>
Complement

If $A$, then $B$  \quad : \quad $ If $A$, then $\neg B$

<table>
<thead>
<tr>
<th>Premise</th>
<th>Conclusion</th>
</tr>
</thead>
</table>
| $A \supset B$ | $A \supset \neg B$ | (not logically valid)  
| $P(A \supset B) = x$ | $P(A \supset \neg B) \in [1-x, 1]$ | (prob. informative)  
| $P(A \land B) = x$ | $P(A \land \neg B) \in [0, 1-x]$ | (prob. informative)  
| $P(B|A) = x$ | $P(\neg B|A) = 1-x$ | (prob. informative)  

| $A \supset B$ | $A \supset \neg B$ | (not logically valid)  
| $P(A \supset B) = .99$ | $P(A \supset \neg B) \in [.01, 1]$ | (pract. non-inform.)  
| $P(A \land B) = .99$ | $P(A \land \neg B) \in [0, .01]$ | (prob. informative)  
| $P(B|A) = .99$ | $P(\neg B|A) = .01$ | (prob. informative)
Complement \( (n_3 + n_4 = 39) \)

\[ A \rightarrow B \quad \therefore \quad A \rightarrow \neg B \]
negated Complement \((n_3 + n_4 = 39)\)
Paradox 3: Monotonicity (Premise strengthening)

“If $A$, then $B$” interpreted as “$A \supset B$”

$\mathcal{P}_1$  If the animal is a bird, then it can fly

$\mathcal{C}$  If the animal is a bird and a penguin, then it can fly

$log. \text{ valid}$

$A \supset B \vdash A \land C \supset B$
Cautious Monotonicity

“If A, then B” interpreted as “$A \supset B$”

$\mathcal{P}_1$ If the animal is a bird, then it can fly
$\mathcal{P}_2$ If the animal is a bird, then it is a penguin

$\mathcal{C}$ If the animal is a bird and a penguin, then it can fly

The second premise “blocks” the conclusion
Monotonicity ($n_3 = 19$)

Frequencies ($n=19$)

$\begin{array}{c}
\text{First Task} & \text{Second Task} \\
\hline
\text{Non-informative} & \text{Yes} & \text{No} \\
\end{array}$

$\begin{array}{c}
\text{Frequencies (n=19)} \\
\end{array}$

$\begin{array}{c}
\text{Non-informative} & \text{Yes} & \text{No} \\
\end{array}$

$\begin{array}{c}
\text{First Task} & \text{Second Task} \\
\hline
\text{A} \rightarrow \text{B} & \therefore \text{C} \land \text{A} \rightarrow \text{B} \\
\varnothing & \therefore \text{A} \land \text{C} \rightarrow \text{B} \\
\end{array}$
negated Monotonicity ($n_3 = 19$)

- Non-informative
- Yes
- No

Frequencies ($n=19$)

First Task: $A \rightarrow B$ \quad \therefore \quad C \land A \rightarrow \neg B$

Second Task: $A \rightarrow B$ \quad \therefore \quad A \land C \rightarrow \neg B$
Cautious Monotonicity ($n_3 = 19$)

- Non-informative
  - First Task: $A \rightarrow B, A \rightarrow C$
  - Second Task: $A \rightarrow C, A \rightarrow B$

- Yes
  - First Task: $A \land C \rightarrow B$

- No
  - First Task: $A \land C \rightarrow B$

- Frequencies ($n=19$)
negated Cautious Monotonicity ($n_3 = 19$)

![Bar chart showing frequencies for non-informative and informative cases in the first and second tasks.]

- **Non-informative**:
  - First Task: $A \implies B, A \implies C$  
  - Second Task: $A \implies C, A \implies B$

- **Informative**:
  - First Task: $A \land C \implies \neg B$  
  - Second Task: $A \land C \implies \neg B$

Legend:
- : First Task
- : Second Task
Conclusions

- Framing human inference in **coherence based probability logic**
  - new predictions (probabilistic *(non-*)informativeness)
  - new experimental paradigms
  - **incomplete** probabilistic knowledge leads to probability-intervals
  - investigating argument forms that differentiate
Conclusions

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- Most participants interpret conditionals as **conditional events**, but...
Conclusions

- Framing human inference in coherence based probability logic
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  - new experimental paradigms
  - incomplete probabilistic knowledge leads to probability-intervals
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- ...differences in interpretations may indicate intra- and interindvidual differences
Conclusions

- Framing human inference in **coherence based probability logic**
  - new predictions (probabilistic (non-)informativeness)
  - new experimental paradigms
  - **incomplete** probabilistic knowledge leads to probability-intervals
  - investigating argument forms that differentiate

- Most participants interpret conditionals as **conditional events**, but...

- ...differences in interpretations may indicate intra- and interindividual differences

- Alternative interpretations, beyond \(||\), \(\supset\), and \(\land\)?
Acknowledgments

- EUROCORES programme LogICCC “The Logic of Causal and Probabilistic Reasoning in Uncertain Environments” (European Science Foundation)
- FWF project “Mental probability logic” (Austrian Research Fonds)

Papers to download:
www.users.sbg.ac.at/~pfeifern/
Appendix
Design Experiment 1

- **Two conditions**: Group 1 \((n_1 = 16)\) and Group 2 \((n_2 = 15)\)

- **Tasks**: Each group 20 tasks (10 arguments affirmative & negated)

- **Group 1**: Five Modus Ponens tasks and five Paradox 1 tasks with varying uncertainties of the categorical premises ("pretty sure" / "absolutely certain", e.g.);
  - Modus Ponens: from \(\text{If } A, \text{ then } B\) and \(A\) infer \(B\)
  - Paradox 1: from \(B\) infer \(\text{If } A, \text{ then } B\)

- **Group 2**: Five Modus Ponens tasks and five Paradox 2 tasks with varying uncertainties of the categorical premises ("pretty sure" / "absolutely certain", e.g.);
  - Modus Ponens: from \(\text{If } A, \text{ then } B\) and \(A\) infer \(B\)
  - Paradox 2: from \(\neg A\) infer \(\text{If } A, \text{ then } B\)
Design Experiment 2

- **Two conditions:** Group 1 ($n_3 = 19$) and Group 2 ($n_4 = 20$)
- **Tasks:** Each group 20 tasks (affirmative & negated)

<table>
<thead>
<tr>
<th>Group 1</th>
<th>informative</th>
<th>not informative</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>COMPLEMENT</td>
<td>IRRELEVANCE</td>
</tr>
<tr>
<td></td>
<td>CAUT. MONOTONICITY I/II</td>
<td>MONOTONICITY I/II</td>
</tr>
<tr>
<td></td>
<td>MODUS PONENS I/II</td>
<td>PARADOX 1 I/II</td>
</tr>
<tr>
<td>Group 2</td>
<td>informative</td>
<td>not informative</td>
</tr>
<tr>
<td></td>
<td>COMPLEMENT</td>
<td>IRRELEVANCE</td>
</tr>
<tr>
<td></td>
<td>MODUS TOLLENS I/II</td>
<td>CONTRAPOS. I/II</td>
</tr>
<tr>
<td></td>
<td>dwr MONOTONICITY I/II</td>
<td>PARADOX 2 I/II</td>
</tr>
</tbody>
</table>
System P: Rationality postulates for nonmonotonic reasoning (Kraus, Lehmann & Magidor, 1990)

Reflexivity (axiom): \( \alpha \models \sim \alpha \)

Left logical equivalence:
\[
\text{from } \models \alpha \equiv \beta \text{ and } \alpha \models \sim \gamma \text{ infer } \beta \models \sim \gamma
\]

Right weakening:
\[
\text{from } \models \alpha \supset \beta \text{ and } \gamma \models \sim \alpha \text{ infer } \gamma \models \sim \beta
\]

Or:
\[
\text{from } \alpha \models \sim \gamma \text{ and } \beta \models \sim \gamma \text{ infer } \alpha \lor \beta \models \sim \gamma
\]

Cut:
\[
\text{from } \alpha \land \beta \models \sim \gamma \text{ and } \alpha \models \sim \beta \text{ infer } \alpha \models \sim \gamma
\]

Cautious monotonicity:
\[
\text{from } \alpha \models \sim \beta \text{ and } \alpha \models \sim \gamma \text{ infer } \alpha \land \beta \models \sim \gamma
\]

And (derived rule): \[
\text{from } \alpha \models \sim \beta \text{ and } \alpha \models \sim \gamma \text{ infer } \alpha \models \sim \beta \land \gamma
\]
System P: Rationality postulates for nonmonotonic reasoning (Kraus, Lehmann & Magidor, 1990)

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from $\models \alpha \equiv \beta$ and $\alpha \models \sim \gamma$ infer $\beta \models \sim \gamma$

Right weakening:
from $\models \alpha \supset \beta$ and $\gamma \models \sim \alpha$ infer $\gamma \models \sim \beta$

Or:
from $\alpha \models \sim \gamma$ and $\beta \models \sim \gamma$ infer $\alpha \lor \beta \models \sim \gamma$

Cut:
from $\alpha \land \beta \models \sim \gamma$ and $\alpha \models \sim \beta$ infer $\alpha \models \sim \gamma$

Cautious monotonicity:
from $\alpha \models \sim \beta$ and $\alpha \models \sim \gamma$ infer $\alpha \land \beta \models \sim \gamma$

And (derived rule): from $\alpha \models \sim \beta$ and $\alpha \models \sim \gamma$ infer $\alpha \models \sim \beta \land \gamma$

$\alpha \models \sim \beta$ is read as If $\alpha$, normally $\beta$
System P: Rationality postulates for nonmonotonic reasoning (Kraus, Lehmann & Magidor, 1990)

Reflexivity (axiom): $\alpha \models \neg \alpha$

Left logical equivalence:
from $\models \alpha \equiv \beta$ and $\alpha \models \neg \gamma$ infer $\beta \models \neg \gamma$

Right weakening:
from $\models \alpha \supset \beta$ and $\gamma \models \neg \alpha$ infer $\gamma \models \neg \beta$

Or: from $\alpha \models \neg \gamma$ and $\beta \models \neg \gamma$ infer $\alpha \lor \beta \models \neg \gamma$

Cut: from $\alpha \land \beta \models \neg \gamma$ and $\alpha \models \neg \beta$ infer $\alpha \models \neg \gamma$

Cautious monotonicity:
from $\alpha \models \neg \beta$ and $\alpha \models \neg \gamma$ infer $\alpha \land \beta \models \neg \gamma$

And (derived rule): from $\alpha \models \neg \beta$ and $\alpha \models \neg \gamma$ infer $\alpha \models \neg \beta \land \gamma$
Semantics for System P

- Normal world semantics (Kraus, Lehmann & Magidor ’90)
- Possibility semantics: \( \alpha \models \beta \iff \Pi(A \land B) > \Pi(A \land \neg B) \)
  (e.g., Benferhat, Dubois & Prade ’97)
  - Empirical support: Da Silva Neves, Bonnefon, & Raufaste (’02), Benferhat, Bonnefon, Da Silva Neves (’05)
- Inhibition nets (Leitgeb ’01, ’04)
- Probability semantics
  - Infinitesimal: \( \alpha \models \beta \iff P(\beta|\alpha) = 1 - \epsilon \)
    (e.g., Adams ’75)
  - Noninfinitesimal: \( \alpha \models \beta \iff P(\beta|\alpha) > .5 \)
    (e.g., Gilio ’02; Biazzo, Gilio, Lukasiewicz, Sanfilippo, ’05)
  - . . .
    - Empirical support: Pfeifer & Kleiter (’03, ’05, ’06)
Modus Ponens \((n_1 + n_2 = 31)\)

Frequencies \((n=31)\):

- Non-informative
- Yes
- No

\[ A \rightarrow B, \quad A \quad \therefore \quad B \]
Modus Ponens ($n_3 = 19$)

- **First Task**: $A \rightarrow B, A \therefore B$
- **Second Task**: $A, A \rightarrow B \therefore B$

<table>
<thead>
<tr>
<th>Frequencies (n=19)</th>
<th>Non-informative</th>
<th>Yes</th>
<th>No</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>2</td>
<td>15</td>
<td>2</td>
</tr>
</tbody>
</table>

![Bar chart showing frequencies for each task]
negated Modus Ponens \( (n_1 + n_2 = 31) \)

\[ A \rightarrow B, \quad A \quad \therefore \quad \neg B \]
negated Modus Ponens ($n_3 = 19$)

Frequencies ($n=19$)

- Non-informative
- Yes
- No

First Task: $A \rightarrow B$, $A$  \therefore \neg B$

Second Task: $A$, $A \rightarrow B$ \therefore \neg B
Modus Tollens ($n_4 = 20$)

- **Modus Tollens**

  Frequencies ($n=20$)

  - $0 \ 5 \ 10 \ 15 \ 20$

  **First Task**

  - $\neg B, A \rightarrow B \therefore \neg A$

  **Second Task**

  - $A \rightarrow B, \neg B \therefore \neg A$

  - $\neg B, A \rightarrow B \therefore \neg A$

  - $A \rightarrow B, \neg B \therefore \neg A$
negated Modus Tollens ($n_4 = 20$)

- **First Task**: $\neg B, A \rightarrow B \therefore A$
- **Second Task**: $A \rightarrow B, \neg B \therefore A$

Frequencies ($n=20$):

- Non-informative: 5
- Yes: 0
- No: 20

Legend:
- ■: First Task
- □: Second Task
Irrelevance \((n_3 + n_4 = 39)\)

\[ A \rightarrow B \quad \therefore \quad A \rightarrow C \]
negated Irrelevance \((n_3 + n_4 = 39)\)

\[\vdash \quad \lnot C\]

\[A \rightarrow B \quad \therefore \quad A \rightarrow \lnot C\]