From the Lockean Thesis to Conditionals

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- $\text{Bel}_P$ is logically closed (in the sense of doxastic logic).
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- *The Lockean thesis* \( LT^{>r} \): \( Bel_P(Y|X) \) iff \( P(Y|X) > r(x?) \geq \frac{1}{2} \).
When is a *conditional* rationally acceptable to an agent?

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- $P$ is a subjective probability measure, $A \rightarrow B$ is a conditional.
- “$\rightarrow$” of the Lockean thesis: If $Bel_P(Y|X)$ then $P(Y|X) > r \geq \frac{1}{2}$.
- $Bel_P(\cdot|\cdot)$ is logically closed (in the sense of AGM 1985 on belief revision).
Can we satisfy all of these desiderata simultaneously?
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If so, how, and does this lead to a unified theory of

- quantitative belief and qualitative belief,
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If so, how, and does this lead to a unified theory of

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(Yes, or so we hope.)

Plan of the talk:

1. The Lockean Thesis and Absolute Belief
2. “→” of the Lockean Thesis and Conditional Belief
3. Epilogue: Solving A Problem Contextually
Let $\mathcal{W}$ be a set of possible worlds, and let $\mathcal{A}$ be an algebra of subsets of $\mathcal{W}$ (propositions) in which an agent is interested at a time. We assume that $\mathcal{A}$ is closed under countable unions ($\sigma$-algebra).

Let $P$ be an agent’s degree-of-belief function at the time.

**P1** (Probability) $P : \mathcal{A} \rightarrow [0, 1]$ is a probability measure on $\mathcal{A}$.

$$P(Y|X) = \frac{P(Y \cap X)}{P(X)}, \text{ when } P(X) > 0.$$  

**P2** (Countable Additivity) If $X_1, X_2, \ldots, X_n, \ldots$ are pairwise disjoint members of $\mathcal{A}$, then

$$P(\bigcup_{n \in \mathbb{N}} X_n) = \sum_{n=1}^{\infty} P(X_n).$$
E.g., a probability measure $P$:

\begin{align*}
\text{A} &: 0.342 \quad \text{B} &: 0.54 \quad \text{C} &: 0.058 \\
\text{A} \cap \text{B} &: 0.018 \\
\text{A} \cap \text{C} &: 0.002 \\
\text{B} \cap \text{C} &: 0.00006 \\
\text{A} \cap \text{B} \cap \text{C} &: 0.03994
\end{align*}

$P$ conditionalized on $C$:

\begin{align*}
\text{A} &: 0 \\
\text{B} &: 0 \\
\text{C} &: 0.1 \\
\text{A} \cap \text{B} &: 0.897 \\
\text{A} \cap \text{C} &: 0.003
\end{align*}
Accordingly, let $Bel$ express an agent’s beliefs.

**B1** (Logical Truth) $Bel(W)$.

**B2** (One Premise Logical Closure) For all $Y, Z \in \mathcal{A}$:
If $Bel(Y)$ and $Y \subseteq Z$, then $Bel(Z)$.

**B3** (Finite Conjunction) For all $Y, Z \in \mathcal{A}$:
If $Bel(Y)$ and $Bel(Z)$, then $Bel(Y \cap Z)$.

**B4** (General Conjunction) For $\mathcal{Y} = \{ Y \in \mathcal{A} | Bel(Y) \}$, $\cap \mathcal{Y}$ is a member of $\mathcal{A}$, and $Bel(\cap \mathcal{Y})$.

It follows: There is a strongest proposition $B_W$, such that $Bel(Y)$ iff $Y \supseteq B_W$. 
In order to spell out under what conditions these postulates are consistent with the Lockean thesis, we will need the following probabilistic concept:

**Definition**

(P-Stability) For all $X \in \mathcal{A}$:

$X$ is $P$-stable$^r$ iff for all $Y \in \mathcal{A}$ with $Y \cap X \neq \emptyset$ and $P(Y) > 0$: $P(X|Y) > r$.

So $P$-stable$^r$ propositions have stably high probabilities under salient suppositions. (Examples: All $X$ with $P(X) = 1$; $X = \emptyset$; and many more!)
In order to spell out under what conditions these postulates are consistent with the Lockean thesis, we will need the following probabilistic concept:

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So $P$-stable $r$ propositions have stably high probabilities under salient suppositions. (Examples: All $X$ with $P(X) = 1$; $X = \emptyset$; and many more!)

Remark: If $X$ is $P$-stable $r$ with $r \in \left[\frac{1}{2}, 1\right)$, then $X$ is $P$-stable $\frac{1}{2}$.

(cf. Skyrms 1977, 1980 on resiliency.)
Then the following representation theorem can be shown:

**Theorem**

Let $Bel$ be a class of ordered pairs of members of a $\sigma$-algebra $\mathcal{A}$, and let $P : \mathcal{A} \rightarrow [0, 1]$. Then the following two statements are equivalent:

1. $P$ and $Bel$ satisfy $P_1$, $B_1$–$B_4$, and $LT \geq P(B_w) > \frac{1}{2}$.

2. $P$ satisfies $P_1$ and there is a (uniquely determined) $X \in \mathcal{A}$, such that
   - $X$ is a non-empty $P$-stable $\frac{1}{2}$ proposition,
   - if $P(X) = 1$ then $X$ is the least member of $\mathcal{A}$ with probability 1, and:

   For all $Y \in \mathcal{A}$:

   $Bel(Y)$ if and only if $Y \supseteq X$

   (and hence, $B_w = X$).

This does not yet presuppose $P_2$. 
Example: Let $W = \{ w_1, \ldots, w_7 \}$, let $P$ be as in the example before.

6. $P(\{ w_7 \}) = 0.00006$
5. $P(\{ w_6 \}) = 0.002$
4. $P(\{ w_5 \}) = 0.018$
3. $P(\{ w_3 \}) = 0.058, P(\{ w_4 \}) = 0.03994$
2. $P(\{ w_2 \}) = 0.342$
1. $P(\{ w_1 \}) = 0.54$

This yields the following $P$-stable sets:

- $\{ w_1, w_2, w_3, w_4, w_5, w_6, w_7 \}$ ($\geq 1.0$)
- $\{ w_1, w_2, w_3, w_4, w_5, w_6 \}$ ($\geq 0.99994$)
- $\{ w_1, w_2, w_3, w_4, w_5 \}$ ($\geq 0.99794$)
- $\{ w_1, w_2, w_3, w_4 \}$ ($\geq 0.97994$)
- $\{ w_1, w_2 \}$ ($\geq 0.882$)
- $\{ w_1 \}$ ($\geq 0.54$)
With P2 one can prove: The class of $P$-stable propositions $X$ in $\mathcal{A}$ with $P(X) < 1$ is well-ordered with respect to the subset relation.

This implies: If there is a non-empty $P$-stable $X$ in $\mathcal{A}$ with $P(X) < 1$ at all, then there is also a least such $X$. 
And \textit{almost all} $P$ over finite $W$ have a least $P$-stable $\frac{1}{2}$ set $X$ with $P(X) < 1!$
And *almost all* $P$ over finite $W$ have a least $P$-stable $\frac{1}{2}$ set $X$ with $P(X) < 1$!

Hence, for *almost all* $P$ there is an $r$, such that there is a $Bel$, where

- **B1–3** Logical closure (with $W$ finite).
- **LT$^r$** For all $X$: $Bel(X)$ iff $P(X) > r$.
- **NT** There is an $X$, such that $Bel(X)$ and $P(X) < 1$. 
Moral:

- Given $P$ and a threshold $r$, the agent’s $Bel$ is determined uniquely by the Lockean thesis.

- $Bel$ is even closed logically iff $Bel$ is given by a $P$-stable set $X$ with $P(X) = r > \frac{1}{2}$.

- So the Lockean thesis and the logical closure of belief are jointly satisfiable as long as the threshold $r$ is co-determined by $P$.

- This almost never collapses into: $Bel(X)$ iff $P(X) = 1$. 
And finally, of course:

- *Lottery Paradox*: Given a uniform measure $P$ on a finite set $W$ of worlds, $W$ is the only $P$-stable set with $r \geq \frac{1}{2}$; so only $W$ is to be believed then.

This makes good sense: the agent’s degrees of belief don’t give her much of a hint of what to believe. *That is why the agent ought to be cautious.*
The following two statements are equivalent:

I. \( P \) and \( \text{Bel} \) satisfy \( P_1 \), the AGM axioms for belief expansion, and \( BP^{r-} \).

II. \( P \) satisfies \( P_1 \), and there is a (uniquely determined) \( X \in \mathcal{A} \), such that \( X \) is a non-empty \( P \)-stable \( r \) proposition, and \( \text{Bel}(\cdot|\cdot) \) is given by \( X \).

The following two statements are equivalent:

I. \( P \) and \( \text{Bel} \) satisfy \( P_1 \)–\( P_2 \), the AGM axioms for belief revision, and \( BP^{r} \).

II. \( P \) satisfies \( P_1 \)–\( P_2 \), and there is a (uniquely determined) chain \( X \) of non-empty \( P \)-stable \( r \) propositions in \( \mathcal{A} \), such that \( \text{Bel}(\cdot|\cdot) \) is given by \( X \).

\( BP^{r-} \) (\( \rightarrow \) of Lockean thesis) For all \( Y \in \mathcal{A} \), s.t. \( P(Y) > 0 \) [and \( Y \cap B_W \neq \emptyset \)]:

For all \( Z \in \mathcal{A} \), if \( \text{Bel}(Z|Y) \), then \( P(Z|Y) > r \).
Example: Let \( W = \{ w_1, \ldots, w_7 \} \), let \( P \) be again as in the example before.

Then if \( Bel(\cdot \mid \cdot) \) satisfies AGM, and if \( P \) and \( Bel(\cdot \mid \cdot) \) jointly satisfy BP\(^1\), then \( Bel(\cdot \mid \cdot) \) must be given by some coarse-graining of the ranking in red below.

Choosing the maximal (most fine-grained) \( Bel(\cdot \mid \cdot) \) yields the following:

- \( Bel(A \land B \mid A) \quad (A \rightarrow A \land B) \)
- \( Bel(A \land B \mid B) \quad (B \rightarrow A \land B) \)
- \( Bel(A \land B \mid A \lor B) \quad (A \lor B \rightarrow A \land B) \)
- \( Bel(A \mid C) \quad (C \rightarrow A) \)
- \( \neg Bel(B \mid C) \quad (C \rightarrow B) \)
- \( Bel(A \mid C \land \neg B) \quad (C \land \neg B \rightarrow A) \)
For three worlds again (and $r = \frac{1}{2}$), the maximal $Bel(\cdot|\cdot)$ as being determined by $P$ and $r$ are given by these rankings:
Moral:

- Given \( P \) and a threshold \( r \), the agent’s \( Bel(\cdot|\cdot) \) is not determined uniquely by the “→” of the Lockean thesis.

- But any such \( Bel(\cdot|\cdot) \) is closed logically iff it is given by a sphere system of \( P \)-stable \( r \) sets.

- Given \( P \) and a threshold \( r \), the agent’s maximal \( Bel(\cdot|\cdot) \) amongst those that satisfy all of our postulates is determined uniquely.

  (And there is always such a unique maximal choice \( Bel'_P \) given a rather weak auxiliary assumption.)
Two remarks:

- $B_1 \rightarrow C_1, \ldots, B_n \rightarrow C_n$ :: $X \rightarrow Y$ is \textit{logically valid} iff for all $P$, $r \geq \frac{1}{2}$ holds:

  - If $\text{Bel}_P(C_1|B_1), \ldots, \text{Bel}_P(C_n|B_n)$
  - then $\text{Bel}_P(Y|X)$.

The resulting logic is exactly Adams’ logic of conditionals again! E.g.:

\[
\frac{X \rightarrow Y, X \rightarrow Z}{X \rightarrow (Y \land Z)} \quad \text{(And)} \quad \frac{X \rightarrow Z, Y \rightarrow Z}{(X \lor Y) \rightarrow Z} \quad \text{(Or)}
\]

\[
\frac{(X \land Y) \rightarrow Z, X \rightarrow Y}{X \rightarrow Z} \quad \text{(Cautious Cut)} \quad \frac{X \rightarrow Y, X \rightarrow Z}{(X \land Y) \rightarrow Z} \quad \text{(Cautious M.)}
\]
Two remarks:

- \( B_1 \rightarrow C_1, \ldots, B_n \rightarrow C_n \therefore X \rightarrow Y \) is logically valid iff for all \( P \), \( r \geq \frac{1}{2} \) holds:

  \[
  \text{If } \quad Bel'_P(C_1|B_1), \ldots, Bel'_P(C_n|B_n) \\
  \text{then } \quad Bel'_P(Y|X).
  \]

The resulting logic is exactly Adams’ logic of conditionals again! E.g.:

- \( X \rightarrow Y, X \rightarrow Z \) \( \frac{X \rightarrow (Y \land Z)}{X \rightarrow (Y \land Z)} \) (And)

- \( X \rightarrow Z, Y \rightarrow Z \) \( \frac{(X \lor Y) \rightarrow Z}{(X \lor Y) \rightarrow Z} \) (Or)

- \( (X \land Y) \rightarrow Z, X \rightarrow Y \) \( \frac{(X \land Y) \rightarrow Z}{X \rightarrow Z} \) (Cautious Cut)

- \( X \rightarrow Y, X \rightarrow Z \) \( \frac{(X \land Y) \rightarrow Z}{(X \land Y) \rightarrow Z} \) (Cautious M.)

- **Conditionalization on Zero Sets:**

  \( P^* \), with \( P^*(Y|X) = P(Y|B_X) \), determines a Popper function.

Epilogue: Solving A Problem Contextually

A challenge to the theory:

- Intuitively, Expansion/Revision can be problematic:

\[
\begin{align*}
&Bel'_{P}(Y_1 \lor Y_2 \lor \ldots \lor Y_n | X), \neg Bel'_{P}(\neg Y_i | X) \\
&\frac{Bel'_{P}(Y_i | Y_i \lor (X \land \neg(Y_1 \lor Y_2 \lor \ldots \lor Y_n)))}{Bel'_{P}(Y_1 \lor Y_2 \lor \ldots \lor Y_n | X)}
\end{align*}
\]
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- Intuitively, Expansion/Revision can be problematic:

\[ Bel'_P(Y_1 \lor Y_2 \lor \ldots \lor Y_n \mid X), \neg Bel'_P(\neg Y_i \mid X) \]
\[ Bel'_P(Y_i \mid Y_i \lor (X \land \neg(Y_1 \lor Y_2 \lor \ldots \lor Y_n))) \]

- Lottery’s revenge: For the same reason, if both \( P \) and \( Bel \) represent the same large finite lottery, then \( P(B_W) \) must be very close to 1!
Epilogue: Solving A Problem Contextually

A challenge to the theory:

- Intuitively, Expansion/Revision can be problematic:

\[
\frac{B_e P(Y_1 \lor Y_2 \lor \ldots \lor Y_n \mid X), \neg B_e P(\neg Y_i \mid X)}{B_e P(Y_i \mid Y_i \lor (X \land \neg(Y_1 \lor Y_2 \lor \ldots \lor Y_n)))}
\]

- Lottery’s revenge: For the same reason, if both $P$ and $\text{Bel}$ represent the same large finite lottery, then $P(B_W)$ must be very close to 1!

In both cases, the solution is to make qualitative belief relativized to partitions:

Possible: $B_e P,\{Z\} (Y_1 \lor Y_2 \lor \ldots \lor Y_n \mid X), \neg B_e P,\{Z'\} (Y_1 \lor Y_2 \lor \ldots \lor Y_n \mid X)$. 