Paradoxical Consequences of Balzer’s and Gähde’s Criteria of Theoreticity. Results of an Application to Ten Scientific Theories

Abstract. It is shown that the criteria of *T*-theoreticity proposed by Balzer and Gähde lead to strongly counterintuitive and in this sense 'paradoxical' results: most of the obviously empirical or at least non-theoretical terms come out as theoretical. This is demonstrated for a lot of theories in different areas. On the way, some improved and some new structuralist theory-reconstructions are given. The conclusion is drawn that the *T*-theoreticity of a term cannot possibly be 'proved' on the basis of the mathematical structure of theory *T* alone (as Gähde and Balzer suggest). Rather, an independent notion of pre-*T*-theoreticity and - more importantly - of empiricity is needed; i.e., "not empirical" and "not pre-*T*-theoretical" are independent, necessary but not sufficient conditions for "*T*-theoretical" (this is also a necessary complement of Sneed's original criterion). Finally it is asked whether the structuralist criterion of *T*-theoreticity complemented by such independent conditions would be a satisfactory answer to 'Putnam's challenge', and the answer again is negative: the criterion is not able to distinguish between empirically contentful and completely contentless (superfluous) 'theoretical' terms.

1. Introduction

In simplified words, Sneed (1971) suggested to characterize a term *t* of theory *T* as *T*-theoretical iff every measurement (determination) of term *t* presupposes *T* to be true for at least some applications. Since 1973, Stegmüller (cf. 1973, p. 33) and other exponents of the 'structuralistic' view have claimed that this new criterion would provide a successful answer to Putnam's famous 'challenge' of his (1962), where he urged the need to give not only a negative characterization of theoretical terms by pointing out their non-empiricity (un-observability), but also to say something positive about their function and nature in science. Nevertheless, as was soon recognized, Sneed's criterion contained a lot of problems. The most important one, which was the decisive motivation for the structuralist philosophers to develop a new criterion, is the following: Sneed's criterion refers to
the class of all measurement methods (for a given term t), but of course, this class is never fully known nor determinable. First, this class contains the subclass of all empirical measurement methods for term t (if there are any) – and regarding the problem of determining this subclass the structuralist philosophers’ criticism of the logical empiricists concerning the vagueness of the notion of the ‘empirical’ (or ‘observational’) seems to apply to the structuralists themselves. Secondly and more importantly for structuralist philosophy, the class of all measurement methods for term t of theory T contains the subclass of all measurement methods of t provided by other theories T’ (≠ T) – and this class is clearly relative to a given historical stage of theory-development. For this reason, Balzer and Moulines (1980, pp. 485–489) have relativized their explication of Sneed’s criterion to those measurement methods of term t of theory T which use the basic concepts of T (and hence are formally sets of partial substructures of potential models of T) and can be found in scientific textbooks at a given stage in the development of science. 5

Because of these and other difficulties the structuralist philosophers of science now advance a new criterion of T-theoreticity. The basic idea of this criterion arose from the dissertation of Gähde (1983). In (1985a) and (1985b) Balzer simplified and modified this criterion in some important respects. Thus the new structuralist criterion exists in two different versions – henceforth called the B(alzer)-criterion and G(ähde)-criterion. 7 However, both criteria have one central property in common: they do not refer any longer to the set of all measurement methods of a given term t of theory T, but only to the set of measurement methods of t within T. The common idea of both criteria is the following (in simplified words): Let M(T) be the set of all models of theory T. Then a term t of theory T is T-theoretical if there exists a specialization B ⊆ M(T) which satisfies certain invariance requirements and yields a measurement method for t up to scale-invariance. Thereby, the main difference between the B- and the G-criterion lies in the fact that the latter one requires in addition, that M(T) should not already yield itself a measurement method for t. First, it is conspicuous that whereas Sneed’s original criterion was formally a universal quantification (for all measurement methods: . . .), both new criteria are now an existential quantification (there exists a measurement method B ⊆ M(T): . . .) (cf. Stegmüller 1986, p. 54–56). Moreover, the question whether there exists such a B ⊆ M(T) meeting
the mentioned invariance requirement depends only on the structure of $M(T)$, i.e., the mathematical structure of the theory – and hence, whether a term $t$ is $T$-theoretical or non-$T$-theoretical is according to this new criterion *provable* within the mathematical structure of $T$, completely independent of pragmatical or contingent circumstances (!). This is repeatedly emphasized by Gähde, Balzer and Stegmüller. Balzer and Stegmüller put much confidence in this new criterion. In fact, Balzer claims that with this new criterion the debate on theoreticity has come for the first time to a lasting result and that this new criterion shows impressively the possibility of progress in philosophy of science. Stegmüller even claims that with this new criterion the discussion on the nature of theoretical terms lasting now almost a half century has come to a relative end. – Let’s see.

Since the B-criterion is much simpler than the G-criterion, and furthermore more suitable for application to any theories of different areas, we will develop our main examples and arguments by reference to the B-criterion. In Section 12 we will then show how our results can be transferred to the G-criterion.

2. THE BALZER-DEFINITION OF T-THEORETICITY

Before giving the precise definition of Balzer-$T$-theoreticity, we must first explain some basic structuralist concepts (in its refined version, following Balzer 1985a, 1985b). We will do that by considering the most important example of structuralist theory-reconstruction, namely CPM = classical particle mechanics – whereas the complicated general definitions and some useful lemmas for understanding them are given in the Appendix.

**DEFINITION 1.** $x$ is a classical particle mechanics with $n$ force-kinds ($n \in \mathbb{N}$) – in short: $x \in M(CPM^n)$, where $M(CPM^n)$ is the class of models of CPM$^n$ – iff there exists a $P$, $T$, $s$, $m$, $f_1$, …, $f_n$ such that

$$x = \langle P; T, \mathbb{R}, \mathbb{R}^3; s, m, f_1, \ldots, f_n \rangle$$

and

1. $P$ is a finite, nonempty set
2. $T$ is a nonempty open interval of $\mathbb{R}$
3. $s: P \times T \to \mathbb{R}^3$ and $s \in C^\infty$
For all $i \in \{1, \ldots, n\}$: $f_i: P \times T \rightarrow \mathbb{R}^3$

(4) $m: P \rightarrow \mathbb{R}^+$

(5) $\forall p \in P \forall t \in T \left( m(p) \cdot s''(p, t) = \sum_{i=1}^{n} f_i(p, t) \right)$

(c.f. Balzer 1985a, p. 19. Note that, here and in the following, $Qx \in X(A)$ is an abbreviation for $Qx(x \in X \rightarrow A)$, where $Q \in \{\exists, \forall\}$.)

Here, $P$ represents the sets of physical objects (particles $p$). Physical units are for simplicity neglected in structuralism; hence the time $T$ is simply represented as an open interval of $\mathbb{R}$, the vector space $\mathbb{R}^3$ over $\mathbb{R}$ represents the euclidean space, the position function $s$ and the force functions $f_i$ are mappings from $P \times T$ into $\mathbb{R}^3$, i.e., for each $p$ and $t$ and $i$ ($1 \leq i \leq n$), $s(p, t)$ and $f_i(p, t)$ are vectors in $\mathbb{R}^3$. $s \in C^\infty$ means that the position function should be infinitely often differentiable with respect to time. The mass function is a time-independent mapping of $P$ into $\mathbb{R}$ such that $\forall p \in P (m(p) > 0)$, which is abbreviated by $m: P \rightarrow \mathbb{R}^+$ ($\mathbb{R}^+$ is the set of positive real numbers). As can be seen, Axioms (1)–(5) only characterize the conceptual net of $x$, each axiom characterizing the nature of one concept, whereas Axiom (6) is Newton's second law. $s''(x, t)$ denotes the second derivative of $s(x, t)$ with respect to time $t$. The set $M(\text{CPM}^n)$, or in short $M$, is simply the class of all models (or structures) $x = (\mathbb{R}; T, \mathbb{R}, \mathbb{R}^3; s, m, f_1, \ldots, f_n)$ satisfying these axioms. Furthermore, the set of potential models $M_p(\text{CPM}^n)$ is defined as the set of all those structures $x$ satisfying Axioms (1)–(5) – i.e., satisfying only those axioms which characterize the conceptual net.

This example shows the most important set-theoretical properties of the sets $M_p$ and $M$ of structuralist theories. (1) For any theory, all structures in $M$ and $M_p$ have the general form $\langle D_1, \ldots, D_k; A_1, \ldots, A_l; r_1, \ldots, r_m \rangle$, where the $D_i$ with $1 \leq i \leq k$, $k \in \mathbb{N}$ are called base sets and represent physical (real) objects, the $A_i$ with $1 \leq i \leq l$, $l \in \mathbb{N}_0$ are called auxiliary base sets and represent sets of mathematical entities – their mathematical structure as well as purely mathematically defined relations and functions (like $+, \cdot$, etc.) are simply presupposed and not explicitly reconstructed. The $r_i$ with $1 \leq i \leq m$, $m \in \mathbb{N}$ are relations of a certain $(k + l)$-type $\tau_i$, i.e., they are subsets of sets which can be constructed out of the $D_1, \ldots, D_k$; $A_1, \ldots, A_l$ by a certain iterative application of the product set operation and the power set operation. In most cases of quantitative theories, the $r_i$ are functions,
which have base sets (plus auxiliary base sets) as their domains and some auxiliary base set as their range. The so called type of a structure is fixed by the numbers \( k, l \) and \( m \) and by the types \( \tau_i \) of the relations. All structures in \( M \) and \( M_p \) are of the same type, in other words, \( M \) and \( M_p \) are subsets of so called typified classes of structures. 

(2) An important property of the structures is that objects of the base sets \( D_i(1 \leq i \leq k) \) are characterized only by the relations \( r_i(1 \leq i \leq m) \) - hence the classes \( M \) and \( M_p \) are closed under isomorphisms with respect to the \( D_i \) and \( r_i \), in which the mathematical sets have to remain identical - so-called canonical transformations (cf. Balzer 1985a, p. 14f). Nonempty sets of structures of the same type closed under such canonical transformations are called species of structures. 

(3) Each of the axioms for \( M_p \) containing some relation characterizes only a single relation in isolation from the others; whereas the additional axioms for \( M \) relate some relations in a non-trivial way (i.e., they are not reducible to a conjunction of single-relation-axioms) (Balzer 1985a, p. 18).

Now, the general structuralist definitions of \( M \) and \( M_p \) identify these sets (for any theory) with species of structures, such that \( M \subset M_p \) and the additional requirements mentioned in (3) are fulfilled. The set-theoretical definitions of these concepts are prima facie complicated and are presented in the Appendix, where also two lemmas (Lemmas 2 and 3) are given which show that if the axioms have the properties informally explained above, then the set-theoretical definitions are indeed satisfied – in particular: if the axioms have certain properties such that they imply no restrictions on the base sets except on their cardinality, then the sets \( M \) and \( M_p \) are indeed species of structures; and if they satisfy the additional requirements mentioned in (3) above, then also the set-theoretical definitions of \( M_p \) and \( M \) (cf. Definition 23 Appendix) are satisfied. With the help of these lemmas it can be easily checked that all theories (and also their specializations used in measurement methods) which I will present in the following sections satisfy the general set-theoretical definitions for species of structures, \( M_p \) and \( M \). (Besides, let us note that these lemmas or a generalization of them would also be important for the structuralist program: to give for each \( M \) and \( M_p \) a proof that it is indeed an \( M \) and \( M_p \) according to the set-theoretical definitions is – although intuitively clear – a complicated matter. Therefore Balzer omits these proofs in (1985a), whereas he gives some in (1985b). With the help of these lemmas, the satisfaction of the set-theoretical definitions can be seen for a large
class of theories by the nature of their axioms – which, last but not least, shows that closer linguistic investigations can be fruitful for the structuralist approach.)

Now, a theory-core according to structuralism is the triple \( \langle M_p, M, Q \rangle \) where \( Q \) is a constraint on \( M_p \) (cf. Balzer, 1985a, p. 23f: \( Q \subseteq \text{Pow}(M_p), \emptyset \in Q, Q \neq \emptyset \)) which says for example in the case of CPM that if several domains \( D_i \) are considered, the objects in the intersection of the \( D_i \) have identical masses. Furthermore, the pair \( \langle \langle M_p, M, Q \rangle, I \rangle \) is called a theory-element (Balzer, 1985a, p. 30f, Stegmüller, 1986, p. 100), where \( I \) is the so-called set of intended applications (Balzer, 1985a, pp. 30–33). Finally, a theory-net is a set of theory-elements ordered by the relation of specialization (Stegmüller, 1986, p. 102). But the problem of theoreticity can be treated with respect to the pair \( \langle M_p, M \rangle \) alone – also Balzer concentrates on such pairs (1985a, p. 37): and in (1985b, p. 132) he even defines a theory as a pair \( \langle M_p, M \rangle \). So we will presuppose from now on that \( \langle \langle M_p, M, Q \rangle, I \rangle \) is some theory-element and concentrate on those logical properties of the pair \( \langle M_p, M \rangle \) which are essential for the B- (or G-) criterion of theoreticity. Furthermore, if the informal axiomatization of \( M \) is given, then \( M_p \) is always the set of structures of the given type satisfying only those axioms which contain at most one relation. Therefore it is enough to present the definition of \( M; M_p \) is obtainable from it.

The basic idea of the B-criterion is this: a relation \( r_i \) of \( T \) is \( T \)-theoretical iff it is determinable by the other relations. Such a determination is also called a measurement of \( r_i \). If \( r_i \) is a function into an auxiliary base set, then scale-invariance must be taken into consideration. Scale-invariances reflect purely conventional elements in the measurement of the values of physical parameters by given arguments. For instance, mass is only measurable up to a constant positive factor – representing the unit mass. The position in space depends on fixing the origin and angle of the system of coordinates and some constant unit interval (e.g., meter) – hence it is measurable up to linear transformations and rotations. Multiplication by a constant positive factor and linear transformations are important examples of scale-invariances, but we will consider some others later on.

Furthermore, a method of measurement is not always given by the theory itself, but often by a specialization of it – i.e., by a subset \( B \subseteq M \) which is characterized by additional axioms or restrictions. Now
Balzer’s basic requirement is that such a specialization is only admissible as a measurement for \( r_i \) if it puts restrictions only on other relations \( r_j \) (\( j \neq i \)) or on the cardinality of the base sets, but not on \( r_i \) itself (otherwise, trivial specializations of \( M \) measuring \( r_i \) would always be possible; cf. Balzer 1985a, p. 142).

Before introducing the definition, let us first note that if \( r_i \) is a relation of a theory with core \( \langle M_p, M \rangle \) (with \( k \) base sets, \( l \) auxiliary base sets), then \( \bar{r}_i = \{ r_i \exists x \in M_p (r_i = pr_{k+l+i}(x)) \} \) is called the corresponding \( i \)-th term of \( T \) (where \( pr_{k+l+i} \) is a projection operator, picking out the \( k + l + i \)-th element of the \( k + l + m \)-tuple, which is the \( i \)-th relation – cf. Balzer, 1985a, p. 11, 1985b, p. 132). Hence, the \( i \)-th term is the set of all \( i \)-th relations of potential models. The notion of “being theoretical” is applied to these terms. \( r_i \in \bar{r}_i \) is called a realization of \( \bar{r}_i \). Note that for simplifying the way of speaking we will call in the following also any \( r_i \) \( T \)-theoretical iff the corresponding \( \bar{r}_i \) is \( T \)-theoretical. Furthermore, if \( M \) is a class of models, \( x \) and \( y \in M \) and \( r_i^x, r_i^y \) – which are the \( i \)-th-relations of \( x \) and \( y \), respectively – are functions into a vector space on an ordered field, then let “\( r_i^x \sim r_i^y \)” express the proposition that \( r_i^x \) and \( r_i^y \) are equivalent with respect to an appropriate scale invariance (which depends on the physical or real nature of these functions). For instance, if “\( \sim \)” is multiplication by a positive constant and the ordered field is \( \mathbb{R} \), then \( r_i^x \sim r_i^y \) means \( \exists k \in \mathbb{R}^+ \forall \alpha (r_i^x(\alpha) = k \cdot r_i^y(\alpha)) \), where \( \alpha \) is a string of variables of the appropriate type. Or if “\( \sim \)” is linear transformation, the ordered field and the vector space are \( \mathbb{R} \) and \( \mathbb{R}^3 \), respectively, then \( r_i^x \sim r_i^y \) means \( \exists k_1 \in \mathbb{R}^+ \exists k_2 \in \mathbb{R}^3 \forall \alpha (r_i^x(\alpha) = k_1 r_i^y(\alpha) + k_2) \) (cf. Balzer 1985a, p. 52f). Formally, every “\( \sim \)” is an equivalence relation. Note furthermore that the case \( r_i^x \sim r_i^y \) also covers the case \( r_i^x = r_i^y \) (it often happens in real examples of theories, that the unit of some relation is fully determined by the given units of other relations).

Furthermore we introduce the following notation. If \( x \in M_p \) (for some \( M_p \)), then \( x_{-i}[r^*] \) denotes the results of the replacement of \( x \)'s \( i \)-th relation by \( r^* \), where \( r^* \) is of the same type as \( r_i \). (E.g., \( \langle D_1, \ldots, A_i; \ r_1, \ldots, r_m \rangle_{-i}[r^*] = \langle D_1, \ldots, A_i; \ r_1, \ldots, r_{i-1}, \ r^*, \ r_i+1, \ldots, r_m \rangle \). Furthermore \( x_{-i} \) denotes the result of the omission of \( x \)'s \( i \)-th relation (with \( x \) as above, \( x_{-i} = \langle D_1, \ldots, A_i; \ r_1, \ldots, r_{i-1}, \ r_{i+1}, \ldots, r_m \rangle \) (cf. Balzer 1985a, p. 37). If \( r_i = a \), then we will also write \( x_{-a} \) for \( x_{-i} \) and furthermore, \( x_{-a,b} \) for \( x_{-a_{-b}} \).

The theoreticity of a term \( \bar{r}_i \) (and correspondingly, of a relation \( r_i \)) is
now defined in three steps:

**DEFINITION 2.** \(B \subseteq M_p\) is an **admissible measurement method** for \(i_i\) iff \(B\) is a species of structures and \(\forall x \in B \forall r, r' (x \in [r] \in B \land x \in [r'] \in B \rightarrow r \equiv r')\) where \(r \equiv r'\) is defined as follows: if \(r, r'\) are functions into a vector space on an ordered field, then \(r \equiv r'\) iff \(r \sim r'\), otherwise \(r \equiv r'\) iff \(r = r'\).

(cf. Balzer, 1985a, p. 38, p. 50f, p. 143f; 1985b, p. 134).\(^{19}\) Hence, if two potential models \(x, y \in B\) are identical with respect to all elements except the \(i\)-th relation, then their \(i\)-th relations are identical – if they are functions into a vector space – up to appropriate scale-transformations. Note that the second condition is equivalent to \(\forall x, y \in B (x_i = y_i \rightarrow r_i \equiv r_i)\). The elements of \(B\) are called measurement-models (cf. Balzer, 1985a, p. 38). If the relations are functions, then the satisfaction of the second condition is usually proved by showing that in the \(B\)-structures the value of \(r_i\) is calculable (up to scale-transformations) by the values of the other functions \(r_j (j \neq i)\) for any given arguments (cf. Balzer, 1985a, p. 38).

**DEFINITION 3.** Let \(B \subseteq M\). \(B\) is called **\(M\)-\(i\)-invariant** iff \(\forall x, y (x \in B \land y \in M \land x_i = y_i \rightarrow y \in B)\).

(cf. Balzer, 1985b, p. 134 and 1985a, p. 143).\(^{20}\) Hence a specialization \(B\) is **\(M\)-\(i\)-invariant** iff: if \(x \in B\) and \(y \in M\) differs from \(x\) only with respect to \(r_i\), then also \(y \in B\). We will write in the following also "**\(M\)-\(r_i\)-invariant"** for "**\(M\)-\(i\)-invariant"**. **\(M\)-\(i\)-invariance** follows generally, if the additional axioms characterizing \(B\) don't contain the relation \(r_i\) (cf. also Balzer, 1985a, p. 142)\(^{21}\) – which we note in the following lemma:

**LEMMA 1.** If \(B \subseteq M\) and the additional axioms characterizing \(B\) don't contain relation \(r_i\), then \(B\) is **\(M\)-\(i\)-invariant**.

**Proof.** If \(x \in B\), and \(y \in M\) differs from \(x\) only with respect to \(r_i\), then the additional axioms for \(B\), which are satisfied by \(x\), must be satisfied by \(y\) as well (by substitution of identicals in the axioms). Hence \(y \in B\).

Now the definition of **\(T\)**-theoreticity\(_B\) (**\(T\)**-theoreticity according to Balzer) is easy:
DEFINITION 4. $\tilde{r}_i$ is $T$-theoretical if there exists a $B \subseteq M$ ($M$ being the class of models of $T$) which is an admissible measurement method for $\tilde{r}_i$ and is $M$-$i$-invariant. Otherwise, $\tilde{r}_i$ is non-$T$-theoretical. (cf. Balzer, 1985a, p. 144; 1985b, p. 134). Depending on the scale invariance which is taken into consideration in the measurement method, we will speak in the following also of "$T$-theoreticity with... invariance ...", where... contains a description of the invariance relation (possibly being also identity – the strongest case of invariance relations).

3. REQUIREMENTS ON A PROPER DEFINITION OF $T$-THEORETICITY

There are some physical (or other) parameters which are obviously empirical, i.e., measurable by empirical means alone (up to conventional elements like scale-transformations). The first requirement is:

(R1) An obviously empirical term should not come out as $T$-theoretical (for some non-trivial and scientifically accepted theory $T$).

Most scientists think that time and position in space are empirical, as well as velocity and other parameters (see below). All these parameters can be measured by empirical operations at least in an approximative way (cf. Note 2) without presupposing some theory except purely conventional or definitorial ‘theories’ characterizing the measurement operation. Now of course, there are philosophers who doubt that and claim that every parameter is at least implicitly ‘theory-loaded’, for instance space is relative to theory of space (a position which I doubt). However, even if one holds the general view of the ‘theory-ladenness’ of all terms, an equivalent of our requirement can be formulated. For, obviously, some theories contain more theoretical terms and stronger (or ‘deeper’) theoretical assumptions than others. We can assume the set of theories partially ordered by a relation of pre-theoreticity, which can be characterized in accordance with Balzer and Mühlhölzer (1982, p. 32) in the following way: theory $T$ is a pre-theory of theory $T'$ iff $T'$ uses all concepts of $T$ whereas $T$ uses less concepts than $T'$. For example, if space is
theoretical in regard to a theory $T$ of space-metric, then $T$ is clearly a pre-theory of CPM. Hence it can be required:

(R2) An obviously pre-$T$-theoretical term should not come out as $T$-theoretical.

We will formulate a third adequacy requirement for $T$-theoreticity in Section 14; in the following we shall concentrate on (R1) and (R2), which concern the adequacy of the distinction $T$-theoretical/non-$T$-theoretical, yielded by the B-criterion. We will show that the B-criterion does not provide an adequate distinction. In order to show this in a convincing way, the examples of theories discussed by us must be 'representative', i.e., we must give a lot of examples. Since we will give on the way some new and some improved structuralist theory-reconstructions, our enterprise has also a value independent from our main problem.

4. PHENOMENOLOGICAL AND KINETIC GAS THEORY

We first reconstruct the phenomenological theory of ideal gases:

DEFINITION 5. $x$ is an ideal gas-phenomenology, $x \in M(IGP)$, iff there exists a $D$, $T$, $k$, $p$, $v$, $n$, $A$ such that

$$x = (D; T, \{k\}, R; p, v, n, A)$$

and

1. $|D| = 1$
2. $T$ is an interval of $R$
3. $k \in R^+$
4. $p: D \times T \to R_0^+$
$v: D \times T \to R_0^+$
$A: D \times T \to R_0^+$
$n: D \times T \to R^+$
5. $\forall x \in D \forall t \in T (p(x, t) \cdot v(x, t) = k \cdot n(x, t) \cdot A(x, t))$

I.e., an ideal gas consists of one object (the gas; Axiom (1)), such that for all times (of the considered interval) a functional relationship between pressure $p$, volume $v$, mole number $n$ and absolute temperature $A$ holds. Note that in explicating $n$ as time-dependent it is allowed for
the gas to have mass exchange during $T$. Furthermore, $p$, $v$ and $A$ can also attain zero values (this is included in the ‘ideal picture’: the extensions of the molecules are considered as negligible, ideally zero; if these molecules don’t move, the values of $p$, $v$ and $A$ become zero). $k$ is the gas constant. Alternatively, we could have omitted $k$ as member of the auxiliary base sets and reformulated Axiom (5) by an existential quantification: $\exists k \in \mathbb{R}^+ \ldots$ But we prefer our version since the gas constant is fully determined if the units of the parameters $p$, $v$, $n$ and $A$ are given. For instance, if $v$ is measured in liters and $p$ in atmospheres, and as always, $A$ in °K (Kelvin) and $n$ in mole numbers, then $k = 0.008205 (\ldots) 1 \cdot \text{atm/°K} \cdot \text{mol}$. In the case of a pure gas (one kind of molecule), $n$ is defined as $m/(M \cdot N_A)$, where $m$ is the mass of the gas, $M$ the mass of one molecule of the gas, and $N_A$ is Avogadro’s number. In the case of a gas mixture, $n$ is defined as $\sum_{i \in I} n_i$, where $I$ is an index set for the gas kinds and $n_i = m_i/(M_i \cdot N_A)$ (cf. Barrow, 1973, p. 9–11). Of course, this definition could be taken up into the formulation of $M_{\text{IGP}}$. (The set $M_{\text{IGP}}$ is clearly characterized by Axioms (1)–(4)).

Now, in phenomenological gas theory, the parameters $p$, $v$, $A$ are all empirical. They can be measured simultaneously by purely empirical means: $v$ simply by a volume scale on the gas container, $p$ by a barometer and $A$ by a thermometer. Balzer (1987) calls this situation of measurement, where all terms are measurable independently by empirical means, ‘isolated measurement’ (isolated from a theory). But we have:

**THEOREM 1.** $\bar{p}$, $\bar{v}$ and $\bar{A}$ are IGP-theoretical with identity-invariance.

**Proof.** First we consider $\bar{p}$: From Axiom (5) it follows that $\forall x \in D \forall t \in T (p(x, t) = k \cdot n(x, t) \cdot A(x, t)/v(x, t))$, if it is presupposed that $v(x, t) > 0$ for all $x \in D$, $t \in T$. We define $B \subset M$ by only this restriction: $B := \{x|x \in M \text{ and } \forall y \in D^x \forall t \in T^x (v^x(y, t) > 0)\}$ (where $D^x$, $T^x$ and $v^x$ are the respective members of model $x$). Obviously, $B$ is $M$-1-invariant ($p$ is the first relation: and consider Lemma 1: in the additional axiom for $B$, $p$ is not contained; i.e., $B$ does not restrict $p$). Because of the above equation (following from Axiom (5)), $B$ is a measurement method for $p$ (without scale-invariance) – i.e., if $x_{-1}[r]$ and $x_{-1}[r'] \in B$, then $r = r'$. Hence $\bar{p}$ is IGP-theoretical with identity-invariance. For $\bar{v}$ we prove it in the same way with the help of
\[ v(x, t) = k \cdot n(x, t) \cdot \mathcal{A}(x, t)/p(x, t) \]
and considering a \( B \) with \( p(x, t) > 0 \); for \( \mathcal{A} \) we use
\[ \mathcal{A}(x, t) = p(x, t) \cdot v(x, t)/(k \cdot n(x, t)) \]
and put \( B = M \) (since \( \forall x \in D \forall t \in T(n(x, t) > 0) \) and \( k > 0 \) hold already in \( M(IGP) \)).

So we have a first strongly counterintuitive result. Balzer mentions the problem of ideal gas law briefly in (1985a, p. 149), but he thinks it is a single exception. We will show in the following by means of many more examples that it seems to be the regular case. – First, it could be conjectured that this problem is only due to the fact that the ideal gas law is an empirical or phenomenological theory, whereas the B-criterion is specifically adapted to deeper theories like CPM. However, this conjecture is not true: if we extend the theory IPG to kinetic gas theory, which contains interrelations between the empirical parameter \( \mathcal{A} \) and the theoretical parameter of mean kinetic energy of the molecules, then the result is just the same:

**Definition 6.** \( x \) is an ideal gas kinetics, \( x \in M(IGK) \), iff there exists a \( D, T, k, N_A, p, v, n, \mathcal{A}, M, \overline{v^2} \) such that
\[
x = \langle D; T, \{ k \}, \{ N_A \}, \mathbb{R}; p, v, n, \mathcal{A}, M, \overline{v^2} \rangle
\]
and

1. \( x_{-N_A, M, \overline{v^2}} \in M(IGP) \)
2. \( N_A \in \mathbb{R}^+ \) is Avogadro’s number \( (6.023 \cdot 10^{23}) \).
3. \( \overline{v^2} : D \times T \to \mathbb{R}^+ \)
4. \( M : D \to \mathbb{R}^+ \)
5. \( \forall x \in D \forall t \in T(\mathcal{A}(x, t) = N_A \cdot M(x) \cdot \overline{v^2}(x, t)/k) \)

(cf. Barrow, 1973, pp. 25–32). \( x_{-N_A, M, \overline{v^2}} \) is the result of the omission of \( N_A, M \) and \( \overline{v^2} \) in \( x \). \( \overline{v^2}(x, t) \) is the mean quadratic velocity of the molecules of gas \( x \) at time \( t \). Note that the word ‘kinetics’ is here used because it is so used in science; of course, IGK is not a pure kinematics, but a simple mechanics, since it contains mechanical terms. By the same proof as above we get:

**Theorem 2.** \( \tilde{p}, \tilde{v} \) and \( \tilde{\mathcal{A}} \) are IGK-theoretical with identity-invariance.

Theory IGK is a macrodescription; of course, it would also be possible to reconstruct it on the microlevel: then \( D = \{ P \} \) where \( P \) is nonempty.
and finite and represents the set of all molecules, \( v: P \times T \rightarrow \mathbb{R}^3 \), and \( \bar{v}^2 \) is defined by \( \bar{v}^2(P, t) = \sum_{p \in P} |v(p, t)|^2 / |P| \). Thus, we would get the theory IGMK (ideal-gas-micro-kinetics). Furthermore, it would also be possible to give a reconstruction of van der Waals’ improved gas law (for an explication of van der Waals’ law within the ‘statement view’ cf. Schurz/Weingartner, 1987, p. 68) – and again, \( p, v, \mathcal{A} \) will come out as WGP-theoretical (WGP = van der Waals’ gas phenomenology). We dispense with giving the detailed reconstruction of these two theories, because the principal facts are clear.

5. LENGTH OF WIRES AND ELECTRIC RESISTANCE

In 1985a (p. 73–75) Balzer presents a chain of measurement models for measuring the electric resistance of wires by Ohm’s law and the amperage of an electric current with one battery (the reconstruction is due to Mühlhölzer, 1981). The point of this chain is that if two wires are given which have cross sections of the same area, such that their resistance is proportional to their length, then the internal resistance and the voltage (‘Urspannung’) of the battery can be measured up to multiplication by a positive constant factor, and with these values the resistance of other wires can be determined by measurement of the current using Ohm’s law. In the following, I will reconstruct an analogue to Balzer’s chain of measurement models consisting of only a single type of models, which I call models of ‘Ohm’s theory with battery and length of wires’. (Alternatively, it would be possible to reconstruct a chain of models and then speak of theoreticity with respect to a class of model-chains.)

DEFINITION 7. \( x \) is a model of Ohm’s theory with battery and length of wires, \( x \in M(\text{OBL}) \), iff there exists a \( D, P, Q, l, r, u, b, i \) such that

\[
D \text{ is a finite, non-empty set} \\
P \subseteq D \\
|Q| = 1
\]
D is a set of wires, and $P$ a distinguished subset of wires of $D$ for which the length $l$ is defined and where length is proportional to resistance ((9)). The set $Q$ contains only one element – the battery $y \in Q$; $b(y)$ is its internal resistance and $u(y)$ its voltage (‘Urspannung’), $r$ is the resistance of the wires in $D$, and $i$ their current in an electric circuit with the battery $y \in Q$. $i$ is presupposed to be non-theoretical (e.g., measured by a magnetic needle). (10) is Ohm’s law.

The point of Balzer’s chain (1985a, p. 75) reformulated within our $M(\Omega B W)$ is that if $|P| > 2$ such that there exists $a, b \in P(l(a) \neq l(b))$, then $b, u$ and $r$ are determined by $l$ and $i$ up to multiplication by a positive constant factor. Hence, by putting $B \subseteq M$ as this subset (which is OBL-$i$-invariant for $i = 2, 3, 4$) $r, b$ and $u$ come out as OBL-theoretical (cf. Balzer, 1985a, p. 72f; it is trivial to transfer Balzer’s proof for the measurability of $r, b$ and $u$ by $l$ and $i$ to our OBL). – But the problem is: length also is OBL-theoretical, since we can reverse the measurement method and measure length by electric resistance for wires with cross section of the same area. (By analogy with the above, the resistances themselves can be measured (up to a positive constant factor) for all wires with the help of Ohm’s law by their currents and the given (different) values of resistance for at least two wires). The trick is that we have simply to define $B$ by the constraint $|P| = |D|$:

**THEOREM 3.** $l$ is OBL-theoretical$_B$ with invariance of multiplication with a positive constant factor.

*Proof.* Let $B \subseteq M$ be defined as $B := \{z \in M \mid |D^z| = |P^z|\}$. $B$ is $M$-1-invariant ($l$ is the 1st relation). From (2) follows $D^z = P^z$ for all $z \in B$. From that and (9) follows $\exists \alpha \in \mathbb{R}^+ \forall x \in D(\hat{l}^z(x) = \alpha \cdot \hat{r}^z(x))$ for all $z \in B$. Hence $l$ is determined in $B$ up to multiplication by a constant positive factor.

Although the proof is indeed trivial, its result is strongly counterintuitive: length is obviously empirical (or at least pre-OBL-theoretical).
The fact that measurement methods are thinkable in which \( l \) can be measured by the given knowledge of theoretical terms – reversing the usual methods – does not change anything about the empirical nature of \( l \). This gives us a first hint that the idea of explicating theoreticity and non-theoreticity purely in terms of internal measurability is misleading.

6. THE HARMONIC OSCILLATOR

Gähde (1983, p. 141; cf. also Stegmüller 1986, p. 220) gives the following reconstruction of a HCPM:

**DEFINITION 8.** \( x \) is a model of CPM with Hooke-force, \( x \in M(\text{HCPM}) \), iff there exists a \( P, T, s, m, f \) such that

\[
\begin{align*}
(1) & \quad x \in M(\text{CPM}) \\
(2) & \quad |P| = 1 \\
(3) & \quad \forall p \in P \exists t \in T(f(p, t) > 0) \\
(4) & \quad \exists k \in \mathbb{R}^+ \exists s_0 \in \mathbb{R}^3 \forall p \in P \forall t \in T(f(p, t) = -k \cdot (s(p, t) - s_0))
\end{align*}
\]

\( x \in M(\text{HCPM}) \) is a CPM with only one force-kind – Hooke’s force \( f(\cdot := f_1) \). (2) restricts the particle set to one object (the oscillating particle) and (3) requires that \( f \) should not always be zero. \( k \) is Hooke’s constant (characterizing the spring). (4) is Hooke’s law (note that it is a vector equation), where \( s_0 \) is the position of the oscillating body at rest (where the spring is unstretched and unexpanded).

For applying HCPM to one-dimensional undamped oscillation two additional restrictions are needed: First, the force should always operate on a straight line, from which follows by (4) that the body oscillates on a straight line. Second, the force values should always be within the Hookean range of the spring. In (1983), p. 58ff (cf. also Stegmüller 1986, p. 219) Gähde gives some restrictions for HCPM if applied to harmonic oscillations, but they seem to me to be much too strong since they already include the detailed sinusoidal position function of the oscillating body, which should come out as derived empirical content of the theory of harmonic oscillation (by solving the differential equation) and should not be included in the axioms.
Hence, our axioms for the specialization $B \subseteq M(\text{HCPM})$ for one-dimensional undamped oscillations will only include those axioms and restrictions which are needed to give the essential description of undamped one-dimensional harmonic oscillation – the rest (including the periodic sinusoidal position function) can be derived.

Now, theory HCPM with the restriction of one-dimensional oscillation can be used for different goals; for instance, the position function can be predicted by initial position and velocity; or mass ratios can be determined from periods (as Gähde 1983 does), or forces can be calculated (from mass and position). Therefore, mass and force are clearly HCPM-theoretical. But the position function also can be calculated from the given knowledge of mass and force function up to the scale transformation of adding a constant – namely $s_0$, the point at rest, which is not theoretically determined, but arbitrary. Therefore, $s$ also is HCPM-theoretical. We show this by defining the restriction of one-dimensional oscillation not – as Gähde does – for the position function, but for the force function, since our $B \subseteq M(\text{HCPM})$ must be $M$-1-invariant.\footnote{This result is not the same as Gähde's, who restricted the position function.}

**THEOREM 4.** $\bar{s}$ is HCPM-theoretical$_B$ with invariance of adding a constant (a subcase of linear transformation).

**Proof.** Let $f_m \in \mathbb{R}^+$. Then we define: $B := \{x \in M(\text{HCPM})|(i) \text{ and } (ii)\}$, where

(i) \[ \exists f^* \in \mathbb{R}^3 \forall p \in P \forall t \in T \exists \mu \in \mathbb{R} \left(f^*(p, t) = \mu \cdot f^* \land |f^*(p, t)| \leq f_m\right) \text{ and} \]

(ii) \[ T^x = \mathbb{R}. \]

In (i), $f_m$ is the maximal absolute value of the force functions $f^*$ in $x \in B$ guaranteeing that the springs in all $x \in B$ behave Hookean – hence, $f_m$ characterizes the set of springs we consider in $B$ (but note that for deriving our result, the second conjunct of (i) is not necessary). In every $x \in B$, $f^*(p, t)$ is a multiple of the vector $f^*$ for all $t \in T$, hence the first conjunct of (i) says that all vectors $f^*(p, t)$ are parallel. From this and Axiom (4) follows for all $x \in B$, $\exists s_0 \in \mathbb{R}^3 \exists k \in \mathbb{R}^+ \exists f^* \in \mathbb{R}^3 \forall p \in P \forall t \in T \exists \mu \in \mathbb{R} (s^x(p, t) = -(1/k) \cdot \mu \cdot f^* + s_0)$; i.e., all position points of $p$ defined by the position vectors $s^x(p, t)$ lie on a straight line. Condition (ii) guarantees that the time interval $T$ contains at least one period of the oscillation (of course, a weaker
formulation of (ii) would be possible). Since the additional conditions for \( B \) do not contain the position function, \( B \) is \( M(\text{HCPM}) \)-1-invariant.

Now consider any model \( x \in B \). Newton's second law \( f(p, t) = m(p) \cdot s''(p, t) \) together with Axiom (4) and the restriction of one-dimensionality yield the linear homogeneous differential equation \( \exists k \in \mathbb{R}^+ \forall p \in \mathbb{P} \forall t \in T: \) \( m(p) \cdot s''(p, t) + k \cdot s(p, t) = 0 \).

Its general solution (including the dependence upon the initial conditions) is: \( \exists \omega_0: P \rightarrow \mathbb{R}^+ \exists s_0 \in \mathbb{R}^3 \forall p \in \mathbb{P} \forall t \in T: \) \( s(p, t) = A(p) \cdot \sin(\omega_0(p) \cdot t + \phi(p)) + s_0 \) (for all \( t \in T \) and the object \( p \in \mathbb{P} \)), where the amplitude \( A(p) \) of \( p \) is given by \( A(p) = (s(p, 0) - s_0)^2 + (s'(p, 0)^2/\omega_0(p))^2 \) \( 1/2 \), (\( s(p, 0) - \) initial position of \( p \), \( s'(p, 0) - \) initial velocity of \( p \), \( \omega_0(p) := 2\pi/\tau(p) - \) angular frequency of \( p \), \( \tau(p) - \) period of \( p \) and the phase \( \phi(p) \) of \( p = \arctan((s(p, 0) - s_0)/s'(p, 0)) \) (cf. Spiegel 1979, p. 87; Zachmann 1977, p. 476–482; Gähde 1983, p. 45–48). Furthermore, \( \omega_0(p) \) is related to \( m(p) \) and \( k \) by \( k = m(p) \cdot \omega_0(p)^2 \). By twice differentiating \( s \) with respect to time we get \( s''(p, t) = -\omega_0(p)^2 \cdot A(p) \cdot \sin(\omega_0(p) \cdot t + \phi(p)) \), hence by Newton's second law, \( f(p, t) = m(p) \cdot s''(p, t) = -m(p) \cdot \omega_0(p)^2 \cdot (s(p, t) - s_0) \) is a sinusoidal periodic function of \( t \). From the given periodic force function we can determine the period \( \tau(p) \) by putting \( \tau(p) = \min\{t' \in \mathbb{R}^+ | \forall p \in \mathbb{P} \forall t \in T: f(p, t) = f(p, t + t') \} \). It holds \( \exists s_0 \in \mathbb{R}^3 \forall p \in \mathbb{P} \forall t \in T: s(x, t) = -f(p, t) \cdot (\tau(p)^2/m(p) \cdot 4\pi^2 + s_0) \) – hence position is determined by force and mass up to adding a constant \( (s_0). \) I.e., for all \( x \in B \): if \( x_{-1}[s], x_{-1}[s'] \in B \), then \( s(x, t) = s'(x, t) + k \) (with \( k \in \mathbb{R}^3 \), for all \( p \in \mathbb{P} \), \( t \in T^x \)).

It is no wonder that HCPM enables a determination of position by mass and force function, but should this be a reason for regarding \( s \) as HCPM-theoretical? We think, it should be not. Furthermore, with a more complicated model of oscillation, which includes also damping forces and is based on events such that time is construed as a function from the set of events into real numbers, it will also be possible to determine the time function by means of the position function – hence within such a model (which we don't reconstruct here because this would be too lengthy), even time would come out as theoretical.

It could be argued against our examples OBL and HCPM that these are specialized theory-elements, whereas the criterion of theoreticity
should be applied to more general (= weaker) theory-elements like CPM itself. But this argument would be ad hoc and not compelling. Firstly, there is no ‘inner’ reason to apply the criterion of theoreticity only to some theory-elements and not to others, e.g., such which are specialized by additional laws (Balzer himself applies the criterion to specialized theory-elements, like CCM, see below). And secondly, if a term is empirical or pre-$T$-theoretical and hence non-$T$-theoretical in some general theory $T$, then this term should of course remain empirical or pre-$T$-theoretical if $T$ is strengthened by additional laws. Hence, even if the theoreticity of the empirical functions could be avoided in a weaker theory $T$, then it would be strongly counterintuitive that these empirical functions get theoretical if $T$ is strengthened by additional laws. – But besides this point: as we will see below, the same counterintuitive results arise also in more general theories, including CPM itself. So in any case, the situation seems to be hopeless.

7. FREQUENCY IN QUANTUM MECHANICS

In order to show that the same problem arises also in more complicated theories of modern physics, we proceed to quantum mechanics of stationary nucleus-electron-systems (atoms, molecules or their ions). By solving the time-independent Schrödinger equation for the given nucleus-electron-system, the energy-levels can be calculated – exactly for one-electron-systems and approximately for many-electron-systems. For the energy-levels the frequency of the (absorption or emission) spectrum of the system can be predicted – and the degree of approximation with the empirically measured frequencies is an important touchstone for the adequacy of the theory (or its respective specialization for the considered system). – Now, frequency in quantum mechanics and quantum chemistry is typically regarded as empirical by scientists. Even if that is philosophically doubted, frequencies are clearly pre-quantum-mechanics-theoretical. But since they can be determined by the theory, they come out as theoretical according to the B-criterion. In order to show this, I will give a reconstruction for the simplest case – for one-nucleus-one-electron-systems (hydrogenium and atomic ions with one electron), for which an exact solution of the Schrödinger equation is possible.33
DEFINITION 9. $x$ is a stationary one-nucleus-one-electron-quantum mechanics, $x \in \text{M}(\text{SQM}_{1,1})$, iff there exists a $P$, $z$, $H$, $\Delta$, $e$, $\nu$ such that

$$x = \langle P; N, \mathbb{R}, \text{PO}, \mathbb{C}, L^2(\text{PO}, \mathbb{C}), D_H; z, H, \Delta, e, \nu \rangle$$

where $\text{PO} := \mathbb{R} \times [0,2\pi] \times [0, 2\pi]$, $L^2(\text{PO}, \mathbb{C}) := \{ f: \text{PO} \to \mathbb{C} | f \text{ is square-integrable} \}$ and:

1. $|P| = 1$
2. $z: P \to \mathbb{N}$
3. $D_H = \{ f \in L^2(\text{PO}, \mathbb{C}) | f \in C^\infty \}$
4. $H: P \times D_H \to L^2(\text{PO}, \mathbb{C})$ and $H$ is linear and hermitean
5. $e: P \to \mathbb{R}^-$
6. $\nu: P \to \mathbb{R}^+$
7. $\forall p \in P \forall f \in D_H \forall (r, \alpha, \beta) \in \text{PO}(H(p, f) = H(p, r, \alpha, \beta) \cdot f)$
   with $H(p, r, \alpha, \beta) := (-1/2r^2) \cdot ((\partial / \partial r) \cdot r^2 \cdot (\partial / \partial r)$
   \begin{align*}
   &+ (1/\sin \alpha) \cdot (\partial / \partial \alpha) \cdot \sin \alpha \cdot (\partial / \partial \alpha) \\
   &+ (1/\sin^2 \beta) \cdot (\partial^2 / \partial \beta^2) ) - z(p)/r.
   \end{align*}
8. Let for all $p \in P$ and $\Psi \in \{ f: P \times \text{PO} \to \mathbb{C} \}$, $\Psi_p$ be defined by
   $\Psi_p: \text{PO} \to \mathbb{C}$ with $\forall (r, \alpha, \beta) \in \text{PO}(\Psi_p(r, \alpha, \beta) = \Psi(p, r, \alpha, \beta))$. Then: $\Delta = \{ \Psi: P \times \text{PO} \to \mathbb{C} | \forall p \in P((i) \text{ and (ii) and (iii))}, \}$, where:
   (i) $\Psi_p \in D_H$
   (ii) $\Psi_p$ is normed, and
   (iii) $\exists e \in \mathbb{R}^- \forall (r, \alpha, \beta) \in \text{PO}(H(p, \Psi_p)(r, \alpha, \beta) = e \cdot \Psi_p(r, \alpha, \beta))$
9. Let $\Psi_p$ be defined as in (8). Then:
   $\forall p \in P(e(p) = \{ e \in \mathbb{R}^- | \exists \Psi \in \Delta \forall (r, \alpha, \beta) \in \text{PO}(H(p, \Psi_p)(r, \alpha, \beta) = e \cdot \Psi_p(r, \alpha, \beta))\}$
10. $\forall p \in P(\nu(p) = \{ \nu \in \mathbb{R}^+ | \exists e_1, e_2 \in e(p)(\nu = + |e_1 - e_2|)\}$

The set $P$ contains only one element: the nucleus-electron-system. (This is a simplification possible in the one-nucleus-one-electron case. If we had introduced electron- and nucleus sets separately, the Hamilton operator would have had to be applied to a more complicated construction of these two base sets representing the system formed by the particles). Next, $\text{SQM}_{1,1}$ contains some new auxiliary base sets: $\text{PO}$ is the euclidean space represented by polar coordinates (this simplifies the wave equations), $\mathbb{C}$ is the set of complex numbers and $L^2(\text{PO}, \mathbb{C})$ is the Hilbert-space of square-integrable functions from $\text{PO}$ into $\mathbb{C}$, representing the mathematical frame for the possible wave functions.
of one-electron-one-nucleus-systems.\textsuperscript{34} $D_H$, the domain of the Hamilton operator $H$, is the subset of all those functions of $L^2(\mathcal{P}_0, \mathbb{C})$ which are infinitely times partially differentiable. In the whole reconstruction we suppose for simplicity \textit{atomic units} (i.e., $\hbar$ is the unit of action (Wirkung) and the charge and mass of the electron are units of charge and mass, respectively – see Kutzelnigg, 1975, p. 69f).\textsuperscript{35} $Z$ is the charge of the nucleus (because of atomic units a natural number). $H$ is the Hamilton operator of the system, formally a mapping from a subset of $L^2(\mathcal{P}_0, \mathbb{C})$ (namely $D_H$) into $L^2(\mathcal{P}_0, \mathbb{C})$. $H$ is presupposed to be linear and hermitean, as all operators in quantum mechanics.\textsuperscript{36} $\Delta$ is the set of wave functions which are eigenfunctions of $H$, representing the possible eigenstates of the electron in the positive centric electric field of the nucleus. The functions of $\Delta$ have as additional argument the set $\mathcal{P}$ containing the one electron-one-nucleus-system, because they reflect properties of the system. For all $\Psi \in \Delta$, $\Psi_p$ represents the pure ‘mathematical’ projection of $\Psi$ on $p \in \mathcal{P}$, and it is a normed element of $D_H$ (condition (i) and (ii))\textsuperscript{37} which is eigenfunction of $H$ (condition (iii)).

$$\int \int \int_{x \in \mathcal{P}_0} \Psi^*(r, \alpha, \beta) \Psi(r, \alpha, \beta) r^2 \, dr \, \sin \alpha \, d\alpha \, d\beta$$

is according to the Kopenhagen interpretation the probability of finding the electron in the space interval $x$, where the nucleus is presupposed to be in the origin of $\mathcal{P}_0$ (of course, we could have introduced the scalar probability density as additional relation of SQM, but we don’t need it for our purpose). The set $\Delta$ is uncountable, since every linear combination of eigenfunctions with the same eigenvalue is also an eigenfunction of $H$ (cf. Kutzelnigg, 1975, p. 78). $\Delta$ has a complete orthonormalized subset $\Delta^*$ which is a basis for $\Delta$ and is countable (since the eigenvalues are discrete), but for simplicity we dispense with taking up this set $\Delta^*$ into the models. $e$ is the function attaching to the system the set of the absolute energy levels of the electron in the respective eigenstates (they are negative real numbers and their absolute values represent the ionization energy). $\nu$ is the function attaching to the system the set of its energy differences between the electron eigenstates, which at the same time represent the
set of (emmission or absorption) frequencies (because of atomic units). In practice, the elements of the countable sets $e$ and $\nu$ are indexed by natural numbers – but again we dispense with that for simplicity.

It is conspicuous that in SQM$_{1,1}$ some terms are definable by others (e.g., $\nu$ by $e$), but this definability does not say very much about physical reality or importance. Clearly, $\nu$ is SQM$_{1,1}$-theoretical. But we get a much stronger result: the quadruple $\langle H, \Delta, e, \nu \rangle$ is completely determined by the other parameters (which shows that even if we would omit $e$ or other terms defining $\nu$, the situation for $\nu$ would not be different).

THEOREM 5. (i) $\bar{\nu}$ is SQM$_{1,1}$-theoretical$_{B}$ with identity-invariance. (ii) $\langle H, \Delta, e, \nu \rangle$ is SQM$_{1,1}$-theoretical$_{B}$ with identity-invariance.

Proof. We choose $B := M(SQM_{1,1})$. $H$ is completely determined by $z$ and $D_{H}$. The Schrödinger equation for one-electron-one-nucleus-systems has an exact solution, which is represented by $\Delta$. Hence $\Delta$ is determined by $z$ and $D_{H}$. $e$ and $\nu$ are determined by $H$ and $\Delta$, hence also by $z$ and $D_{H}$, which proves (ii), from which (i) follows. □

8. CLASSICAL COLLISION MECHANICS

Balzer proves the adequacy of his criterion for three cases of theories: CCM = classical collision mechanics, CPM = classical particle mechanics, and EE = exchange economics (Balzer, 1985a, p. 144–150; Balzer, 1985b, p. 135–144). I will argue that in the case of CCM his result depends upon a very ‘weak’ formulation of CCM and in the case of CPM it depends upon a physically implausible ad hoc-restriction. I will not consider EE here in detail, but I will finally show that the same problem exists for a pretheory of EE: the theory of preferences and utility. First, let us consider CCM.

Balzer’s proof for CCM in (1985a, b) is based on the reconstruction of CCM in Balzer/Mühlhölzer (1982). Here they investigate the possibilities of determining mass ratios by collision experiments. On the other hand, it is not possible in their reconstruction of CCM to calculate final velocities (after collision) from masses and initial velocities (1982, p. 26). Therefore, in their models velocity comes out non-theoretical, in conformity with their expectation. – But physics textbooks on collision usually give methods of calculating final velocities from masses and initial velocities. This is possible because here
certain special cases are considered. One such special case is plastic collision, where the colliding particles adhere after collision. In this case, final velocities are determinable from initial velocities. By the additional introduction of the force function, i.e., embedding PCCM (plastic collision mechanics) into CPM, \( \nu \) is determinable up to a constant.

**DEFINITION 10.** \( x \) is a CPM with actio-reactio for plastic collision, \( x \in M(PACPM) \), iff there exists a \( P, T, t_1, t_2, s, m, f \) such that

\[
x = \langle P; T, [t_1, t_2[, \mathbb{R}, \mathbb{R}^3; s, m, f \rangle
\]

and

\[
(1) \quad x_{t_1, t_2} \in M(CPM^1) \\
(2) \quad [t_1, t_2[ \text{ is an open interval of } T \\
(3) \quad \forall p \in P \forall t \in T (t \in [t_1, t_2[ \implies f(p, t) = 0) \\
(4) \quad \forall p, p' \in P \forall t \geq t_2 (s(p, t) = s(p', t)) \\
(5) \quad \forall t \in T \left( \sum_{p \in P} f(p, t) = 0 \right)
\]

\( [t_1, t_2[ \) is the interval within which the collision happens. (3) and (4) give the empirical conditions for describing the collision: before and after the collision, the particle move force-free; and after time \( t_2 \) the particles adhere, i.e., in the idealized picture of point masses, they have the same position in space. Note that from condition (4) it follows that \( \forall p, p' \in P \forall t \geq t_2 (s'(p, t) = s'(p', t)) \), i.e., all particles also have the same velocity after collision. Our description is similar to Gähde’s description of an inelastic collision (Gähde, 1983, p. 35; Stegmüller, 1986, p. 219), but there is a difference concerning condition (4). \( ^{39} \) (5) is the actio-reactio-principle in its ‘derived’ form. Note that Balzer/Mühlhölderz (1982, p. 30) introduce for each pair of particles the force with which the first acts on the second (cf. p. 30, D7(6), D8(3)). In our reconstruction \( f(p, t) \) is the sum of all forces on \( p \) from the other particles \( q \neq p \). Of course, we could introduce \( f^*: P \times P \times T \to \mathbb{R}^3 \) and define \( f(p) = \sum_{q \in P} f^*(p, q, t) \) – then, our principle (5) can be derived from the actio-reactio-principle \( f^*(p, q, t) = -f^*(q, p, t) \) (with the special case \( f^*(p, p, t) = -f^*(p, p, t) = 0 \)).

Now, as is well known, under certain conditions final velocities can
be calculated from initial conditions and masses by the conservation of momentum, which follows from the integration of (5) with respect to time. But that is not our problem – we want to determine velocity by given forces and masses. Here we do not even need conservation of momentum: we only need to integrate the force function for each particle – division through mass then yields its velocity relative to the final velocity. From the condition that the final velocity is equal for all particles we can determine the velocity of all particles up to adding a constant – the final velocity. The more complicated formulation of Theorem 6 is due to the fact that strictly speaking we cannot directly apply PACPM-theoreticity to \( v \), since \( v \) – although definable by \( s \) – is not a relation of PACPM. But we can define a variant of PACPM, \( \text{PACPM}_v \), which is formulated with \( v \) instead of \( s \).

**THEOREM 6.** (a) Define \( V_p \in P \forall t \in T(v(p, t) := s'(p, t)) \). Then \( \tilde{v} \) (velocity) is PACPM-measurable by \( \tilde{m} \) and \( \tilde{f} \) up to adding a constant.

(b) Let \( \text{PACPM}_v \) be the variant of PACPM where in the models \( s \) is replaced by \( v \), in Axiom (3) of CPM\(^1\) and in Axiom (4) of PACPM \( s \) is replaced by \( v \), and in Axiom (5) of CPM\(^1\) \( s'' \) is replaced by \( v' \) (\( \text{PACPM}_v \) is the formulation of PACPM with \( v \) instead of \( s \)). Then: \( \tilde{v} \) is \( \text{PACPM}_v \)-theoretical\(_B\) with invariance of adding a constant.

**Proof.** For each \( p \in P \), \( \int_0^t f(p, t) dt = m(p) \cdot \int_0^t s''(p, t) dt = m(p) \cdot \int_0^t dv(p, t)/dt dt = m(p) \cdot \int_0^t dv(p, t) = m(p) \cdot (v(p, t2) - v(p, t)) = m(p) \cdot \Delta(p, t) \), where \( \Delta(p, t) := v(p, t2) - v(p, t) \) is the velocity difference between \( p \)'s velocity to \( t2 \) and \( p \)'s velocity to \( t \). (Note that this is a vector equation, and note that the integral \( \int_0^t f(p, t) dt \) is a function of its lower bound, which is defined for \( t < t2 \) as well as for \( t \geq t2 \). Moreover, for all \( t \geq t2 \) \( \Delta(p, t) = 0 \) follows from our Axiom (3)).

Hence, for all \( p \in P \), \( t \in T \), \( \Delta(p, t) = (\int_0^t f(p, t) dt)/m(p) \) is determined by the mass and force functions. Furthermore, from (4) follows \( \exists u \in \mathbb{R}^3 \forall p \in P(v(p, t2) := s'(p, t2) = u) \), where \( u \) is the final velocity, equal for all particles. Hence \( \exists u \in \mathbb{R}^3 \forall p \in P \forall t \in T(v(p, t) = -\Delta(p, t) + u) \).

This shows that \( v \) is determined by \( m \) and \( f \) up to adding a constant, which is part (a) of the theorem. Putting \( B = M(\text{PACPM}_v) \) we get part (b) by the same consideration. \( \square \)

The scale invariance of adding a constant for \( v \) has an important physical meaning: it is *Galilean invariance* of classical physics. It reflects the fact that there is no possible method for determining
absolute velocities; what really can be measured—empirically or theoretically—are only relative velocities, i.e., velocities relative to an inertial system—a reference system of coordinates moving with constant velocity. The corresponding Galilean invariance for the position function is yielded by integrating once more with respect to time: 

\[ s^*(p, t) = s(p, t) + v_0 t + s_0 \]

where \( v_0 \) is the relative velocity between the two inertial systems (starred and unstarred), and \( s_0 \) is the relative distance between the two systems at \( t = 0 \). In our PACPM, \( s \) is also determinable up to Galilean transformations. Note that this follows not trivially by integrating one more time—the relative distances between the particles must be calculated from the condition (4), according to which for all \( t \geq t_2 \) all particles have the same position:

**THEOREM 7.** \( \tilde{s} \) is PACPM-theoretical with Galilean invariance.

Proof. For each \( p \in P \),

\[ \int_{t_2}^{t} v(p, t) \, dt = \int_{t_2}^{t} \frac{ds(p, t)}{dt} \, dt = \int_{t_2}^{t} ds(p, t) = s(p, t_2) - s(p, t) \]

is the position difference of the particle \( p \) between time \( t_2 \) and time \( t \). By the equation \( v(p, t) = -\Delta(p, t) + u \) of Theorem 6, we obtain

\[ \int_{t_2}^{t} (u - \Delta(p, t)) \, dt = u \cdot (t_2 - t) - \int_{t_2}^{t} \Delta(p, t) \, dt. \]

From Axiom (4) follows that \( \exists s_2 \in R^3 \forall p \in P(s(p, t_2) = s_2) \)—i.e., at time \( t_2 \), all particles have the same position \( s_2 \). From this and the above follows:

\[ \exists s_2 \in R^3 \exists u \in R^3 \forall p \in P \forall t \in T: \]

\[ s(p, t) = s_2 - u \cdot (t_2 - t) + \int_{t_2}^{t} \Delta(p, t) \, dt. \]

Let us put \( s^*(p, t) := \int_{t_2}^{t} \Delta(p, t) \cdot dt. \) Since \( \int_{t_2}^{t} \Delta(p, t) \, dt = (\int_{t_2}^{t} f(p, t) \, dt \, dt)/m(p) \),

\[ s^*(p, t) \]

is determined by force and mass functions. We get \( \exists u \in R^3 \exists s_2 \in R^3 \forall p \in PVt \in T(s(p, t) = s^*(p, t) + u \cdot t + (s_2 - u \cdot t_2)) \), where \( (s_2 - u \cdot t_2) \) is a constant \( \in R^3 \). This shows that \( s \) is determined in PACPM up to Galilean invariance: if \( x \) and \( y \) are models of \( B := M(\text{PACPM}) \) agreeing in \( f \) and \( m \), then \( \exists k_1, k_2 \in R^3 \forall p \in PVt \in T(s^*(p, t) = s^y(p, t) + k_1 t + k_2). \)

We have noted that for determining \( v \) and \( s \) up to Galilean invariance, we did not need conservation of momentum. Conservation of momentum puts an additional restriction on the velocity differences calculated by forces, namely \( \sum_{p \in P} (m(p) \cdot \Delta(p, t_i)) = 0 \), from which under certain conditions (Balzer/Mühlhölzer, 1982, p. 28f), mass ratios can be calculated from velocity differences. But for the determination of velocity and position from masses and forces this is not needed.

This gives reason to conjecture that this determination is also possible in CPM.
9. Classical Particle Mechanics and Galilean Invariance

Remember Definition 1 of CPM. First let us speak about acceleration \( a(p, t) := s''(p, t) \). Obviously, it should be an empirical or at least non-CPM-theoretical term. But it can be determined without any scale invariance by mass and forces:

**Theorem 8.** (a) Define \( \forall p \in P \forall t \in T, a(p, t) := s''(p, t) \). Then \( \tilde{a} \) (acceleration) is CPM\(^n\)-measurable with identity-invariance by mass and force terms. (b) Let CPM\( ^n_B \) be the variant of CPM\( ^n \) where \( s \) is replaced by \( a \) in the structures and in Axiom (3), and \( s'' \) is replaced by \( a \) in Axiom (6). Then: \( \tilde{a} \) is CPM\( ^n_B \)-theoretical with identity-invariance.

**Proof.** Let \( B = M(\text{CPM}^n) \). By Newton's second law, \( \forall p \in P \forall t \in T, s''(p, t) = \left( \frac{1}{m(p)} \right) g \cdot f_s(p, t) \). Hence, acceleration is fully determined by mass and force functions, from which (a) and (b) follows. \( \square \)

Newton's second law claims for each \( p \in P, t \in T: m(p) \cdot s''(p, t) = \sum_{i=1}^{n} f_i(p, t) \). Hence, for any model \( x \in M(\text{CPM}^n) \) with \( x^{-1}[s] \) and \( x^{-1}[s^*] \in M(\text{CPM}^n) \), it follows that \( \forall p \in P \forall t \in T, s''(p, t) = s^*''(p, t) \) (since mass and force functions are identical in \( x^{-1}[s] \) and \( x^{-1}[s^*] \)). In 1985a (p. 51) Balzer argues that from this it would follow that position is determined by mass and force up to Galilean transformations, because from the above fact it would follow by integrating twice with respect to time that \( \exists k_1, k_2 \in \mathbb{R}^3 \forall p \in P \forall t \in T, s(p, t) = s^*(p, t) + k_1 \cdot t + k_2 \). But this argument is erroneous. What really follows is only \( \forall p \in P \exists k_1, k_2 \in \mathbb{R}^3 \forall t \in T, s(p, t) = s^*(p, t) + k_1 \cdot t + k_2 \). In other words, the integration constants \( k_1 \) and \( k_2 \) can be different for each particle, which means that CPM neither determines the relative velocities nor the relative distances at time zero between the particles (i.e., formally the existential quantifier has to stand behind, and not before \( \forall p \)). So, the measurement of position up to Galilean transformation is not such a trivial matter as Balzer thinks. But anyway it is possible, namely by additional specializations, which determine the relative velocities and distances of the particles. First, there is a trivial specialization of CPM which is CPM-invariant: simply by putting \( |P| = 1 \):

**Theorem 9.** \( \tilde{s} \) is CPM\(^n\)-theoretical with Galilean invariance.
Furthermore: \( \tilde{\sigma} \) is \( \text{CPM}^n \)-theoretical with Galilean invariance, where \( \text{CPM}^n \) is defined as in Theorem 6.

**Proof.** Choose \( B = \{ x \in M(\text{CPM}^n) \mid |P^x| = 1 \} \). \( B \) is \( \text{CPM}^n \)-1-invariant (Lemma 1). By integrating Newton’s second law twice with respect to time, we get \( \forall p \in P \exists k_1, k_2 \in \mathbb{R}^3 \forall t \in T(s(p, t) = s^*(p, t) + k_1 \cdot t + k_2) \) (where \( s^*(p, t) \) such that \( s^{**}(p, t) = (\sum_{i=1}^n f_i(p, t))/m(p) \) and \( s^*(p, t) \) is determined by mass and force functions). Since \( |P| = 1 \), from this follows \( \exists k_1, k_2 \in \mathbb{R}^3 \forall p \in P \forall t \in T(\ldots) \). Hence \( B \) is a measurement method for \( \tilde{s} \) with Galilean invariance. The same considerations (integrating once with respect to time) prove the result for \( \tilde{v} \).

This shows that even for the most important ‘paradigm theory’, namely CPM, the \( B \)-criterion leads to the counterintuitive result that intuitively empirical terms come out as theoretical. – It may be asked whether \( s \) and \( v \) are measurable by \( f \) and \( m \) in CPM-models with more than one particle also. The answer is: yes – if one investigates specializations of CPM with additional information and applies the theoreticity-criterion to these specializations. For instance, our PACPM or HCPM are such specializations of CPM in which additional information enables one to determine relative velocities and distances. In HCPM, \( s \) is even determinable up to a constant, since here the inertial system is already fixed by the spring, which is implicitly assumed to have velocity zero. (Note that these specializations are not \( \text{CPM} \)-1-invariant, hence the criterion for theoreticity has to be applied to these specializations of CPM. But as already argued above, there is no reason against applying the criterion of theoreticity to more specialized theory-elements like HCPM etc.) Another specialization of CPM is a CPM with gravitational forces between the particles:

**DEFINITION 11.** \( x \) is a CPM with gravitational forces, \( x \in M(\text{GCPM}) \), iff there exists a \( P, T, \gamma, s, m \), and a family \((f_q)_{q \in P}\) such that

\[
 x = \langle P; T, \mathbb{R}, \mathbb{R}^3, \{ \gamma \}; s, m, (f_q)_{q \in P} \rangle
\]

and

1. \( x_{-(\gamma)} \in M(\text{CPM}^{[P]}) \) and \( \gamma \in \mathbb{R}^+ \)
2. \( \forall p \in P \forall q \in P: f_q(p, t) = \)
if \( p \neq q \):
\[
- \gamma \cdot m(p) \cdot m(q) \cdot (s(p, t) - s(q, t)) / |(s(p, t) - s(q, t)) |^3
\]
if \( p = q \):
\[
0
\]
This is in essence the same as Moulines' reconstruction of Scheibe's analysis of a Newtonian gravitational system (cf. Stegmüller, 1986, p. 248; note that since units are presupposed as given, \( \gamma \) is the same constant in all \( x \in M(\text{GCPM}) \) -- hence we introduce \{\( \gamma \)\} as an auxiliary base set and do not existentially quantify upon \( \gamma \) in Axiom (2), as it is done in Stegmüller, 1986, p. 248). We list explicitly the forces: \( f_q(p, t) \) is the force of particle \( q \) upon particle \( p \) at time \( t \). Hence, \( (f_q)_{q \in P} \) can be regarded as a family of \( |P| \) force kinds with one particle argument \( p \). Axiom (2) is the gravitational law -- note that it is a vector equation. Note further, that the actio-reactio-principle \( f_q(p, t) = -f_p(q, t) \) follows from our Axiom (2). (Of course, an equivalent possibility would have been to construct \( f \) as a function \( f: P \times P \times T \rightarrow \mathbb{R}^3 \)). Usually, GCPM is used to determine the set of possible position functions from given masses (by solving the differential equation). For our purpose we use it to calculate the actual position function from given forces and masses up to Galilean transformations: for each time \( t \), the relative distances between two particles \( p \) and \( q \), \( s(p, t) - s(q, t) \), can be calculated from given \( f_q(p, t) \) and masses. By integrating Newton's second law twice with respect to time for one particle, the position function is determined up to Galilean transformation for all particles.

**THEOREM 10.** \( \tilde{s} \) is GCPM-\( B \)-theoretical with Galilean invariance.

**Proof.** From the gravitational law follows (\( \forall p \in P \forall t \in T : s(p, t) - s(q, t) = (\gamma \cdot m(p) \cdot m(q) / |f_q(p, t)|)^{1/2} \cdot (-f_q(p, t) / |f_q(p, t)|) \)). The first bracket \((\ldots)^{1/2}\) gives the absolute value of the relative distance between \( p \) and \( q \) at \( t \), and the second bracket gives its direction in \( \mathbb{R}^3 \). Let us write \( \Delta(p, q, t) \) for \( s(p, t) - s(q, t) \); \( \Delta(p, q, t) \) is determined by force and mass functions. Now, we take an arbitrary \( a \in P \): by integrating of Newton's second law twice with respect to time we get (\( \exists k_1, k_2 \in \mathbb{R}^3 \forall t \in T : s(a, t) = (1/m(a)) \cdot \int \int \sum_{q \in P} f_q(a, t) \, dt \, dt = (1/m(a)) \cdot S(a, t) + k_1 \cdot t + k_2 \), with \( S(a, t) \) such that \( S''(a, t) = \sum_{q \in P} f_q(a, t) \) and \( S(a, t) \) is determined by mass and force functions (cf. Note 40). Since for each \( p \in P \), \( t \in T \): \( s(p, t) = s(a, t) + \Delta(p, a, t) \), we get by combination with the above \( \exists k_1, k_2 \in \mathbb{R}^3 \forall p \in P \forall t \in T (s(p, t) = S(a, t) + \Delta(p, a, t) + k_1 \cdot t + k_2) \). Hence, \( \tilde{s} \) is GCPM-measurable by \( \tilde{m} \) and \( \tilde{f}_q \) up to Galilean transformation. In other
words, let \( B = M(GCPM^a) \): if \( x \) and \( y \in B \) agree in mass and force functions, then \( s^y \) is a Galilean transformation of \( s^x \).

10. BALZER ON GALILEAN INVARIANCE

As already noted, Balzer recognizes – although by an erroneous argument – that the position function is measurable in CPM or in specializations of CPM with Galilean invariance. Balzer argues that Galilean invariance has to be excluded from scale-invariance. For which reasons? On page 51 of his (1985a), he mentions that not just any scale invariance may be admitted because this would trivialize the criterion for measurability and theoreticity. This is certainly true: if scale invariance may be just any equivalence relation on the model set, then every relation \( r_i \) will be trivially measurable. But then – indeed surprisingly – Balzer mentions Galilean invariance as such an example of ‘just any’ invariance. Here he conflicts with basic physical intuition. Galilean invariance is not at all ‘just any’ equivalence relation, but one of the most fundamental principles of classical physics! And what this principle states is a scale invariance. As already noted above, a scale invariance reflects the fact that some elements in measuring physical parameters like space and velocity are just arbitrary and conventional. It took a long time in the history of science to find out that only relative velocities are measurable whereas the absolute value of velocity is arbitrary, a matter of convention of fixing the inertial system. In other words: it is arbitrary to assume that the velocity of our reference system of coordinates is zero: everything – all measurable events – will behave in the same way if it is a constant different from zero. This is nothing but a scale invariance – in the same way as the fixing of the origin of the Euclidean space is conventional and, hence, a scale invariance. Therefore: Galilean invariance is the proper scale-invariance for \( s \) and \( v \) in classical physics.

Balzer wants to regard only multiplication by a constant and linear transformation as scale invariance, whereas Galilean invariance should be excluded. As he and Stegmüller (1986, p. 175) think, Galilean invariance is a physical invariance, but not a scale invariance. But in general, there is no sharp distinction between physical invariances and scale invariances – on the contrary: scale invariances reflect exactly
those elements in the measurement of parameters, which are conventional, i.e., not determinable because of physical invariances. Since Galilean invariance reflects the conventional elements in the measurement of $v$ and $s$, it is the appropriate scale invariance. Let us give two additional arguments. First: For Balzer, scale-invariances are only multiplications with a constant and linear transformations — hence, he defines scale-invariances mathematically (1985a, p. 52). According to Balzer, then, Galilean invariance must be a legitimate scale invariance for velocity, because it amounts to adding a constant (a special case of linear transformation). But from this, by integrating with respect to time, the Galilean invariance for position can be derived. This shows that scale-invariance cannot be mathematically defined. It is a physical matter, and not a purely mathematical one. Second: If Balzer and Stegmüller would claim, that the scale-invariance of position is finer than Galilean invariance — the latter being only a theoretical invariance of CPM, but not a scale invariance — then this would imply, that $s$ is measurable in some pre-theory of CPM or in a purely empirical way finer than with Galilean invariance. But position can, strictly speaking, never be measured finer than up to Galilean transformation; so again, Galilean invariance is the proper scale invariance for position.

Balzer gives another argument against Galilean invariance as a scale invariance: he says that the relativization of $s$ and $v$ on a reference system (i.e., inertial system) — which is necessary for determining $s$ and $v$ uniquely — is not possible in CPM, since CPM does not speak about reference systems. But it is no problem, to give a reconstruction of CPM with inertial systems — here it is:

DEFINITION 12. $x$ is a classical particle mechanics with inertial systems, $x \in M(CPM_i)$, iff there exists a $P$, $T$, $s$, $m$, $f_1, \ldots, f_n$ such that

$$x = \langle P; I, T, R, R^3; s, m, f_1, \ldots, f_n \rangle$$

and

1. $I = R^3 \times R^3$
2. $P$ is nonempty and finite
3. $T$ is an open interval of $R$
4. $s: I \times P \times T \rightarrow R^3$ and $s \in C^\infty$
(5) \( m: P \rightarrow \mathbb{R}^+ \)

(6) \( \forall j \in \{1, \ldots, n\}: f_j: I \times P \times T \rightarrow \mathbb{R}^3 \)

(7) \( \forall i \in I \forall p \in P \forall t \in T (s(i, p, t) = s((0, 0), p, t) - pr_1(i) \cdot t - pr_2(i)) \)

(8) \( \forall i \in I \forall p \in P \forall t \in T \left( \sum_{j=1}^{n} f_j(i, p, t) = m(p) \cdot s''(i, p, t) \right) \)

\( I \) is the set of all inertial systems, with each \( i \in I \) defined as a pair \( \langle v_i \cdot s_i \rangle \) where \( v_i \) is the velocity of system \( i \) relative to the distinguished system \( (0, 0) \) and \( s_i \) is the position of the origin of \( i \) at time zero relative to the origin of \( (0, 0) \). Functions \( s \) and \( f_j \) have now as an additional argument the set \( I \). Axiom (7) gives the Galilean transformations between all inertial systems and the distinguished system \( s_0 = (0, 0) \) (\( pr_1 \) is the projection operator, i.e., \( pr_1(i) = v_i, pr_2(i) = s_i \)).\(^{46}\)

From (7), the Galilean transformations between all systems can be derived as \( s(j, p, t) = s(i, p, t) - (pr_1(j) - pr_1(i)) \cdot t - (pr_2(j) - pr_2(i)) \).

(8) is Newton's second law. From CPM\(^n\) clearly follows:

**THEOREM 11.** \( \sum_{j=1}^{n} f_j(i, p, t) \) is \( I \)-invariant, i.e., \( \forall i, i' \in I \forall p \in P \forall t \in T (\sum_{j=1}^{n} f_j(i, p, t) = \sum_{j=1}^{n} f_j(i', p, t)) \).

**Proof.** \( \sum_{j=1}^{n} f_j(i, p, t) = m(p) \cdot s''(i, p, t) = m(p) \cdot \left( d^2(s((0, 0), p, t) - v_i \cdot t - s_i)/dt^2 \right) = m(p) \cdot s''((0, 0), p, t) \) (for all \( i \in I, p \in P, t \in T \)). \( \square \)

Furthermore, in CPM\(^n\) for each particle \( p \) a certain reference system – call it its eigensystem \( e(p) \) – can be defined in the following way: Let \( e(p) \) be that inertial system which has velocity and position of \( p \) in \( (0, 0) \) at time zero. From this follows that velocity and position of \( p \) in \( e(p) \) at time zero are both zero. The addition of the function \( e \) makes \( s \) and \( v \) theoretical with identity-invariance.

**DEFINITION 13.** \( x \in M(\text{CPM}_I^p, e) \) iff there exists a \( P, T, s, m, f_1, \ldots, f_n, e \) such that

\[ x = \langle P; I, T, \mathbb{R}, \mathbb{R}^3; s, m, f_1, \ldots, f_n, e \rangle \]

and

(1) \( x - e \in M(\text{CPM}_I^p) \)

(2) \( e: P \rightarrow I \)

(3) \( \forall p \in P (s'((0, 0), p, 0) = pr_1(e(p)) \land s((0, 0), p, 0) = pr_2(e(p))) \)

**THEOREM 12.** \( \bar{s} \) is CPM\(^n\)\(_{I,e}\)-theoretical with identity-invariance.
Proof. It holds for each particle \( p \in P \): \( s'(e(p), p, 0) = s'((0, 0), p, 0) - pr_1(e(p)) = 0 \), and \( s(e(p), p, 0) = s((0, 0), p, 0) - pr_1(e(p)) \cdot 0 - pr_2(e(p)) = pr_2(e(p)) - pr_2(e(p)) = 0 \).
Hence, the velocity and position of \( p \) in \( e(p) \) at time zero are zero. So the position function is uniquely determined in \( e(p) \) as \( s(e(p), p, t) = \int_0^t \int_0^s s''(p, t) \, dt \, ds \), where \( s''(p, t) \) is determined by mass and force functions and is \( I \)-invariant (so the first argument place can be omitted).
By this, the position function for every \( i \in I \) is determined by \( s(i, p, t) = s(e(p), p, t) - (pr_1(i) - pr_1(e(p))) \cdot t - (pr_2(i) - pr_2(e(p))) \).
So, by the functions \( m, f_i \) and \( e \), the position function (and hence also the velocity function) is fully determined. By putting \( B = M(\text{CPM}_{\mu,e}^I) \) it follows that \( s \) is \( \text{CPM}_{\mu,e}^I \)-theoretical with identity-invariance. \( \square \)

11. Theory of Preferences and Utility

Utilities are usually measured by preferences. Sneed has given a structuralist reconstruction of Jeffrey's decision theory, reported in Stegmüller (1986, p. 395-409):

**Definition 14.** \( x \) is a model of Jeffrey's decision theory, \( x \in M(\text{Jeff}) \), iff

\[
x = (B, \leq_p, P, U)
\]
such that

1. \( B := (A, \land, \lor, \neg, t, f) \) is a Boolean algebra
2. \( \leq_p \subseteq A \times A \)
3. \( P: A \to \mathbb{R} \)
4. \( U: A \to \mathbb{R} \)
5. \( \leq_p \) is reflexive, connected and transitive in \( A \)
6. \( P(t) = 1; \forall x \in A(P(x) \geq 0); \forall x \in A \forall y \in A((x \land y) = f \to P(x \land y) = P(x) + P(y)) \)
7. \( \forall x \in A \forall y \in A(x \land y = f \to U(x \land y) = U(x) \cdot P(x) + U(y) \cdot P(y)) \)
8. \( \forall x \in A \forall y \in A(U(x) \leq U(y) \; \text{iff} \; x \leq_p y) \)

(cf. Stegmüller, 1986, p. 397f, 400f). \( A \) is the set of all possible propositions (\( t \) verum, \( f \) falsum), and the preference relation \( \leq_p \) puts on \( A \) a quasi-ordering. \( P \) is the probability function with standard axioms in (6), \( U \) is the utility function; from (7) the usual formula for
the expected utility can be derived. (8) is the central axiom connecting preferences and utilities. Now, intuitively, in Jeff preferences are empirical and hence non-Jeff-theoretical, whereas utility is Jeff-theoretical. Indeed, utility is measurable in a certain Jeff-invariant specialization of Jeff up to so called Gödel-Bolker-transformations (Stegmüller, 1986, p. 404-411), and is therefore Jeff-theoretical. But the problem is: preferences come out as Jeff-theoretical too - they can be trivially determined by the utility function.

**THEOREM 13.** \( \preceq_p \) is Jeff-theoretical$_B$.

**Proof.** Let \( B = M(\text{Jeff}), \) \( x, y \in B \) with \( x \preceq_p y = y \preceq_p \). \( \forall \) \( b, b' \in A: b \preceq_p b' \) iff \( U^x(b) \leq U^x(b') \) iff \( U^y(b) \leq U^y(b) \) iff \( b \preceq_p b' \). \( \square \)

### 12. Applications to Gähde's Criterion

**12.1. Gähde's Definition of T-theoreticity**

Gähde's criterion of \( T \)-theoreticity rests on the following idea: term \( r_i \) is \( T \)-theoretical if there exists a subset \( r^* \) of \( T \)'s relations such that \( r_i \) is a member of \( r^* \) and in \( T \) alone the members of \( r^* \) are not measurable by the other relations of \( T \), but are measurable in some ‘admissible’ specialization of \( T \). Gähde's original definition of \( T \)-theoreticity runs over five pages (1983, pp. 130-135). We restate it in the following much shorter but equivalent way:

**DEFINITION 15.** Let \( T \) be a theory, \( \{r_1, \ldots, r_m\} \) be the set of its relations, and \( M \) the set of its models. Furthermore we use the following abbreviations: Let \( \mu \subseteq \{1, \ldots, m\}, \) \( x \in M \) and \( y \in M \). Then \( r_\mu := \{r_i | i \in \mu\} \) (hence \( r_\mu \) is some subset of \( T \)'s relations); \( x_\mu \) denotes the result of the omission of all relations \( r_i \in r_\mu \) in \( x \); \( M_\mu, \mu := \{x_\mu | x \in M_\mu\} \) is the set of all \( \mu \)-partial potential models'; and \( r^*_\mu = r^*_\mu \) denotes the proposition that for all \( r_i \in r_\mu \), \( r^*_i = r^*_i \) holds, where \( \equiv \) means equivalence with respect to the appropriate scale invariance, as defined in Definition 2. Finally let \( x \sim_T y \) denote the proposition that models \( x \) and \( y \) are equivalent with respect to the “invariance requirements associated with \( T \)” (1983, p. 133). In the case of CPM, \( \sim_T \) represents Galilean invariance. Then:

\( r_i \) is \( T \)-theoretical$_G$ iff there exists a \( \mu \subseteq \{1, \ldots, m\} \) with \( i \in \mu \)
and

(1) \( \forall x \in M_{\mu-\mu} \exists y, z \in M(x = z_{-\mu} \land \neg (r_{\mu}^y \equiv r_{\mu}^z)) \)

(2) There exists a (proper) specialization \( B \subset M \) such that

(2a) \( B \) is admissible with respect to \( \tau \), i.e., \( \forall x \in B \forall y \in M(x \sim \tau y \to y \in B) \), and

(2b) \( \exists x \in B \forall y \in B(x_{-\mu} = y_{-\mu} \to r_{\mu}^x \equiv r_{\mu}^y) \).

Let us give a short comparison of Gähde's with Balzer's criterion.

**Common properties:** (1) In both criteria the \( T \)-theoreticity of a term depends on the mathematical structure of the theory alone – which is the central locus of our criticism in Section 13. (2) In both criteria the only consideration relevant for the \( T \)-theoreticity of a term is its measurability with the help of other terms under certain invariance requirements – which opens them both to the criticism in Section 14.

**Differences:** (1) Gähde evaluates the question of measurability not with respect to a single relation \( r_i \), but with respect to a whole subset of relations \( r_\mu \subseteq \{r_1, \ldots, r_m\} \), such that \( r_i \in r_\mu \). He requires that all elements of \( r_\mu \) should be measurable by the other relations in some admissible specialization. (2) In addition, Gähde requires that \( r_\mu \) should *not* be measurable in \( T \) alone (a requirement which has no counterpart in Balzer's criterion). Hence Gähde's criterion presupposes a suitable identification of the basic element of a theory and its distinction from specialized elements. (3) Gähde's notion of 'admissible specialization' is weaker than Balzer's notion of '\( M-i \)-invariant specialization' (of Definition 3). Furthermore, for theories different from CPM or other basic physical theories it is often very hard to say what the 'invariance requirements associated with the theory' are. (4) Gähde's condition (2b) is a weaker concept of 'measurement method' than Balzer's Definition 2, since the latter requires \( \forall x, y \in B(...) \), where "(...)" is the matrix of (2b) of Definition 15.

In spite of these differences we will show in the following by somewhat modified constructions, that our examples and their paradoxical consequences, namely that intuitively empirical or non-\( T \)-theoretical terms come out as \( T \)-theoretical, apply also to Gähde's criterion.48

12.2. IGP

When applying Gähde's criterion to the ideal gas phenomenology IGP
of Definition 5 we first note that already $M(IGP)$ itself allows a determination of a single relation by the other relations, in the sense of Gähde's 'weak measurability'. If we choose for example $r_\mu = \{p\}$, then condition (1) of Definition 15 is violated, since there exists a $x \in M_{p,-\mu}$, namely any $x \in M_{p,-\mu}$ with $\forall d \in D^x \forall t \in T^x(v(d, t) > 0)$, such that $\forall y, z \in M$ with $y_{-\mu} = z_{-\mu} = x$, $p_y = p^z$ holds (by the same proof as that of Theorem 1). Nevertheless we could demonstrate $p$ as IGP-theoretical if we could show that $p$ is a member of a set $r_\mu$ which contains more than one relation and is not determinable already in $M$ itself, but in an admissible specialization. Indeed, such a specialization exists.

THEOREM 14. $\bar{p}$ and $\bar{v}$ are IGP-theoretical.

Proof. We choose $r_\mu = \{p, v\}$. Take any $x = (D; T, \{k\}, \mathbb{R}; n, s^x) \in M_{p,-\mu}$ satisfying Axioms (1)-(4) of $M(IGP)$. Obviously, there exists a $p, p^*, v, v^*$ (satisfying Axiom (4) of $M(IGP)$) such that neither $p \equiv p^*$ nor $v \equiv v^*$, i.e., neither $p$ and $p^*$ nor $v$ and $v^*$ are identical up to multiplication with a positive constant, and $\forall d \in D^x \forall t \in T^x(p(d, t) \cdot v(d, t) = p^*(d, t) \cdot v^*(d, t) = k \cdot n^x(d, t) \cdot s^x(d, t))$ holds, i.e., Axiom (5) is satisfied by $p$ and $v$ as well as $p^*$ and $v^*$. By putting $y = (D^x; T^x\{k\}, \mathbb{R}; p, v, n^x, s^x)$ and $z = (D^x; T^x, \{k\}, \mathbb{R}; p^*, v^*, n^x, s^x)$ we see that condition (1) of Definition 15 is satisfied. For $B$ we choose the subset of IGP-models describing one-atomic ideal gases under adiabatic conditions, which means that the gas molecules consists only of one atom and that no heat exchange between the gas and its environment exists. These models satisfy the additional equation $p(d, t)(v(d, t))^a = b$, where $a \in \mathbb{R}^+$ is the same constant for all ideal one-atomic adiabatic gases and $b$ only depends on the mole number (cf. Barrow, 1973, pp. 142-145, 148-151). Hence we define $B := \{x \in M(IGP) | \exists \beta: \mathbb{R} \to \mathbb{R} \forall d \in D^x \forall t \in T^x(p^x(d, t) \cdot (v^x(d, t))^a = \beta(n^x(d, t)))\}$, where $a \in \mathbb{R}^+$. $B$ is lawlike and satisfies all basic physical invariance principles relevant for IGP, and hence it satisfies condition (2a) of Definition 15. We choose now an $x \in B$ with constant mole number, i.e., $\exists c \in \mathbb{R}^+ \forall d \in D^x \forall t \in T^x(n^x(d, t) = c)$. Let $b := \beta(c)$. Then for all $y \in B$ with $y_{-p,v} = x_{-p,v}$ it holds $(\forall d \in D^y \forall t \in T^y): p_y(d, t) \cdot v_y(d, t) = k \cdot c \cdot s^y(d, t)$ and $p^y(d, t) \cdot (v^y(d, t))^a = b$. By these two equations, $p^y$ and $v^y$ are fully determined up to identity-invariance (elementary calculation yields
\[ v^\gamma(d, t) = \exp((\ln(k \cdot c \cdot \mathcal{A}^\gamma(d, t))/b)/(1 - a)) \]; \( p^\gamma(d, t) \) is then determined by \((v^\gamma(d, t))^a \) and \( b \). Hence condition (2b) is satisfied. □

12.3. **OBL**

Ohm's theory with battery and length of wires is already a specialized theory-element, whereas Gähde's criterion should be applied preferable to basic theory-elements. Let us define theory OBW – *Ohms theory with battery and wires* – simply by:

**DEFINITION 16.** \( M(\text{OBW}) := \{x := \langle D, P, Q; \mathbb{R}; l, r, u, b, i \rangle | x \text{satisfies Axioms (1)-(8) and (10) of } M(\text{OBL})\} \).

Then we get the result:

**THEOREM 15.** \( \bar{I} \) is OBW-theoretical with invariance of multiplication with a positive constant factor.

*Proof.* We choose \( r^\mu = \{l, i\} \). Clearly, every \( \langle D, P, Q; \mathbb{R}; r, u, b \rangle \in M(\text{OBW}) \) can be extended by functions \( l, l', i \) and \( i' \) to models \( x, x' \in M(\text{OBW}) \), such that \( l \) and \( l' \) are not equivalent up to scale invariance; hence condition (1) is satisfied. We define \( B := \{x \in M(\text{OBW}) | x \text{satisfies in addition Axiom (9) of } M(\text{OBL}) \text{ and } |D|^x = |P|^x\} \). \( B \) is lawlike and violates no basic invariance requirement associated with OBW, so it satisfies condition (2a). In all \( x \in B \), \( i \) and \( l \) are determined up to \( = \) by \( r, v \) and \( b \) (for \( i \): Axiom 10; for \( l \): Theorem 3), so condition (2b) holds. □

12.4. **CPM**

As already mentioned, Gähde's criterion is especially fitted to the theory CPM and its specializations. Here CPM\( ^n \) is the basic theory-element, so Gähde's criterion should only be applied to CPM\( ^n \), and not to more special theory-elements like HCPM, PACPM or GCPM. The relation \( \sim \) associated with CPM\( ^n \) is Galilean invariance. As argued in Section 11, the appropriate scale invariance for the position function \( s \) is also Galilean invariance. There are two ways to prove that \( s \) is CPM\( ^n \)-theoretical\( _G \). The first way is to choose \( r^\mu = \{s\} \). Condition (1) of Gähde's criterion requires that in no \( x \in M(\text{CPM}^n) \) \( s \) is determined by the other relations up to Galilean invariance. Now we know from Theorem 9 (and the arguments in the paragraph above)
that only in the special case of a one-particle-CPM\textsuperscript{a}, \(s\) is determined by the other relations up to Galilean invariance, and hence that only for this special case condition (1) is not satisfied. This case is in some sense trivial, because only relative distances (between different particles or space-time-points) have physical reality. If we define the (non-trivial) basic theory-element of classical more-particle-mechanics (with \(n\) force-kinds), in short CMPM\textsuperscript{b} by:

\[ M(CMPM^n) := \{x \in M(\text{CPM}^n) | |P^x| > 1\}, \]

- then by choosing \(r_\mu = \{s\}\), \(s\) comes out as CMPM\textsuperscript{b}-theoretical\textsubscript{G}:

**THEOREM 16.** \(s\) is CMPM\textsuperscript{b}-theoretical\textsubscript{G} with Galilean-invariance (by choosing \(r_\mu = \{s\}\)).

**Proof.** Let \(r_\mu = \{s\}\). By the same proof as for Theorem 9 we get for all \(x \in M(CMPM^n)\): \(\forall p \in P^x \exists k_1, k_2 \in \mathbb{R}^3 \forall t \in T^x \langle 2 \rangle (s^x(p, t) = s^*x(p, t) + k_1 \cdot t + k_2\rangle\), where \(s^*x(p, t)\) is determined by \(\sum_{x=1}^{n} f^i(p, t)\) and \(m^i(p)\) (cf. Note 40). Take any \(z := \langle P; T, R, R^3; m, f_1, \ldots, f_n\rangle \in M(CMPM^n)\), and define \(s_1(p, t) := s^*x(p, t)\) for all \(p \in P^z\). Since \(|P^z| > 1\), there exists a \(p_1\) and \(p_2 \in P\) with \(p_1 \neq p_2\). Take four pairwise different constants \(k_1, k_2, k_3, k_4 \in \mathbb{R}^3\) and define \(s_2(p_1, t) := s^*x(p_1, t) + k_1 \cdot t + k_2\), and \(s_2(p_2, t) := s^*x(p_2, t) + k_3 \cdot t + k_4\), and for all \(p \in P^z\) with \(p \neq p_1\) and \(p \neq p_2\) let \(s_2(p, t) = s_1(p, t)\). Then \(s_1\) and \(s_2\) are not Galilean-equivalent; hence by choosing \(x := \langle P^z; T^z, R, R^3; s_1, m^z, f_1^z, \ldots, f_n^z\rangle\) and \(y := \langle P^z; T^z, R, R^3; s_2, m^z, f_1^z, \ldots, f_n^z\rangle\) condition (1) of Definition 15 is satisfied (alphanumeric change of \(x\) and \(z\)). To satisfy condition (2), we consider (for example) our theory GCPM (CPM with gravitational forces), i.e., we put \(B := \{x \in M(CMPM^n) | x \in M(GCPM^n)\}\). \(B\) is a Galilean-invariant specialization, and \(s\) is determined in every \(x \in B\) (and hence in some \(x \in B\)) up to Galilean-invariance, as the proof of Theorem 10 shows; so condition (2) is satisfied.

The second way to prove \(s\) as CMPM\textsuperscript{b}-theoretical\textsubscript{G} is to choose \(r_\mu = \{s, m\}\) and to use HCPM as a specialization with the additional restriction of the same spring in each model. In this specialization \(m\) and \(s\) are simultaneously determined by \(f\) (\(m\) with identity invariance, \(s\) with Galilean invariance). Let us furthermore define the general theory CPM - classical particle mechanics with any number of force kinds by:
DEFINITION 18. \( M(CPM) := \bigcup_{n \in \mathbb{N}} M(CPM^n) \)

(which corresponds to Sneed's reconstruction of CPM, cf. Note 13).

Then we get the following strong result:

THEOREM 17. \( \mathcal{S} \) is CPM-theoretical with Galilean-invariance (by choosing \( r_\mu = \{s, m\} \)).

Proof. Let \( r_\mu = \{s, m\} \). Obviously, every \( x = \langle P; T; \mathbb{R}, \mathbb{R}^3; f_1, \ldots, f_n \rangle \in M(CPM)_{p,-\mu} \) (for every \( n \in \mathbb{N} \)) can be extended by relations \( m \) and \( s \) in different, non-scale-equivalent ways to models of \( CPM^n \), so condition (1) is satisfied.\(^{52} \) We define \( B \) as follows: Let \( f_m \in \mathbb{R}^+ \) and \( k \in \mathbb{R}^+ \). Then

\[
B := \{ x \in M(CPM) | x \text{ satisfies conditions (1)-(3) of } M(HCPM) \text{ (Definition 8) and (4): } \exists t_0, s_0 \in \mathbb{R}^3 \forall p \in P \forall t \in T(f(p, t) = -k \cdot (s(p, t) - (s_0 + v_0 \cdot t)) \text{ and (i) and (ii) in the proof of Theorem (4))} \}.
\]

Note that in the definition of \( B \) we had to write Axiom (4) now in a somewhat different form (compared to Definition 8): \( k \) is now fixed for all \( x \in B \), and the replacement of \( 's_0' \) by \( '(s_0 + v_0 \cdot t)' \) is necessary for making \( B \) Galilean-invariant.\(^{53} \) Since \( B \) is Galilean-invariant, \( B \) satisfies condition (2a) of Definition 15. Take any \( x \in B \). By the same argument as in the proof of Theorem 4 we see that the period \( \tau^x(p) \) is determined by the given periodic force function \( f^x(p, t) \). Because of \( m^x(p) = k \cdot \frac{\tau^x(p)}{4 \pi^2} \) and because \( k \) is the same constant for all \( y \in B \), \( m \) is determined by the force relation in \( B \) with identity invariance, i.e., \( \forall x, y \in B(x_{-s,m} = y_{-s,m} \rightarrow m^x = m^y) \). As shown like in the proof of Theorem 4, \( s^x \) is determined by \( m^x \) and \( f^x \) up to the addition of \( (s_0 + v_0 \cdot t) \); hence \( s^x \) is determined by \( m^x \) and \( f^x \) up to Galilean invariance. Since \( m^x \) itself is determined by \( f^x \) up to identity-invariance, we see that condition (2b) is satisfied (for every \( x \in B \), and hence for some \( x \in B \)).

Let us finally note without comment how Gähde's criterion can be applied to \( CPM^n_{I,e} \) of Definition 13:

THEOREM 18. Let the 'core-theory' \( CPM^n_{I,e} \) be defined by
\[ M(CPM_{t,e}^*) = \{ x \in M(CPM_{t,e}) \mid x \text{ satisfies only Axioms (1) and (2) of } M(CPM_{t,e}) \}. \] Then: \( \bar{s} \) is \( CPM_{t,e}^* \)-theoretical\( _G \) with identity-invariance.

**Proof.** Put \( r_\mu = \{ s \} \). Obviously, \( s \) is not determined in \( M(CPM_{t,e}^*) \), so condition (1) is satisfied. We let \( B := M(CPM_{t,e}) \). The Galilean-invariance of \( B \) is already implicitly contained in the definition of the inertial systems. Furthermore, \( B \) satisfies also condition (2b), as the proof of Theorem 12 shows.

12.5. \( SQM_{1,1} \)

Our stationary one-nucleus-one electron-quantum-mechanics of Section 7 is a very special theory-element. Let us choose as a more general theory-element the theory of one-nucleus-many-electron-quantum-mechanics, \( SQM_1 \), defined by:

**DEFINITION 19**

\[
M(SQM_1) := \{ x := (P, N, R, PO, C, L^2(PO^n, C), D_H, z, H, e, v) \mid n \in \mathbb{N} \text{ and } x \}
\]
satisfies Axioms (1)--(6), (8)--(10) of \( M(SQM_{1,1}) \) except that in Axioms (3), (4), (8) and (9), \( \text{"PO"} \) has to be replaced by \( \text{"PO}^n\)\( \), and \( \text{"r, } \alpha, \beta \text{"} \) by \( \text{"r}_1, \ldots, r_n, \alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n\).\)

(Where \( L^2(PO^n, C) := \{ f: PO^n \rightarrow C \mid f \text{ is square-integrable} \} \).)

Hence \( SQM_1 \) contains the Hamilton operator for one-nucleus-many-electron-systems (with \( 3n \) coordinates for \( n \) electrons) without giving its specific mathematical form. We get:

**THEOREM 19.** \( \tilde{\nu} \) is \( SQM_1 \)-theoretical\( _G \) with identity-invariance.

**Proof.** We choose \( r_\mu = \{ H, \Delta, e, \nu \} \). Since \( \Delta, e \) and \( \nu \) are determined by \( D_H, z \) and \( H \), and since \( H \) is not specified in \( M(SQM_1) \), every \( x \in M(SQM_1)_{r_\mu} \) can be extended in several different non-scale-equivalent ways by relations \( H, \Delta, e \) and \( \nu \), thus condition (1) is satisfied. We define \( B := M(SQM_{1,1}) \subseteq M(SQM_1) \). \( B \) is an admissible specialization, since it satisfies the basic invariance requirements which are contained in the linearity and hermiticity of \( H \). So condition (2a) is satisfied. That condition (2b) is satisfied is seen by the same proof as for Theorem 5. \( \square \)
12.6. **Jeff**

By choosing $r_\mu = \{\leq_p, U\}$ and defining a very special $B$, in which the utility of a proposition is identical with its probability – call it the special set of *uncertainty-reduction-utility-models* – we can show the desired results:\(^5^4\)

**THEOREM 20.** $\leq_p$ is Jeff-theoretical.

*Proof.* Let $r_\mu = \{\leq_p, U\}$. Obviously, every $(B, P) \in M(Jeff)_p,\mu$ can be extended by relation $\leq_p$ and function $U$ to a Jeff-model in different (not scale equivalent) ways, so condition (1) is satisfied. We define $B := \{x \in M(Jeff) | \forall b \in A^x(U(x) = P(x))\}$. $B$ is a lawlike specialization, and there is no invariance requirement associated with Jeff (if there is any) violated by $B$; so condition (2a) is satisfied. Trivially, also condition (2b) is satisfied. \(\square\)

13. **MAIN CONCLUSION:** T-THEORETICITY OR NON-T-THEORETICITY CANNOT BE DECIDED BY INTERNAL MEASURABILITY WITHIN $T$

We have shown for a large and representative set of examples, that empirical or non-$T$-theoretical relations are or can be theory-internally measurable and hence come out as $T$-theoretical according to the B-criterion as well as to the G-criterion. Note that from this follows that in many of our examples of theories, e.g., in CPM" and its specializations, even all terms come out as theoretical$_B$ as well as theoretical$_G$, so B-theoreticity and G-theoreticity do not distinguish here between any terms. There seem to be no chances of avoiding these counterintuitive results by some further technical modifications of these criteria. The deeper philosophical reason is clear; whether a term is empirical or non-theoretical, depends on certain facts outside of the theory. For instance, whether a term can be determined in some pre-theory of $T$ is a fact lying outside of the theory $T$. But most importantly, whether a term is *empirical* and hence non-theoretical depends on the *contingent* fact of the nature of human observation and experience: what we can directly observe – for instance positions of bodies, lengths, colours etc. – is a *contingent* fact and never determinable by theory-internal measurability. For instance, if humans would
have sensors for magnetic fields (as some fishes have), then the term of magnetic field strength would be empirical – independently of the mathematical structure of electromagnetic theory. To decide about the question of empiricity vs. theoreticity by a logical proof within the theory is a basically misleading idea. Of course, some terms of a theory can be determined by other ones: but this does not say anything about their empirical (non-theoretical) or theoretical nature. Theoretical terms can be calculated by empirical ones, but conversely, empirical terms are also calculable by theoretical terms.

So our main conclusion is: necessary conditions for an adequate criterion of theoreticity are two notions independent from $T$-internal measurability: (1) a notion of pre-$T$-theoreticity and (2) a notion of empiricity. In other words, “to be not pre-$T$-theoretical and not empirical” is a necessary and irreducible conjunct of the definiens of “to be $T$-theoretical” (for any theory $T$). In some sense, this conclusion suggests a return back to Sneed’s original criterion, since in this criterion $T$-theoreticity is relativized to the set of all (empirical, pre-$T$-theoretical or $T$-theoretical) measurements. But also in Sneed’s criterion no explication of “empirical” and “pre-$T$-theoretical” is given; so the mentioned independent conditions are also a necessary complement of Sneed’s original criterion. – In Section 3, we have presented an explication of “pre-$T$-theoreticity” due to Balzer/Mühlhölzer. This explication is very sketchy; but anyway, the chances for a satisfactory explication are not so bad here. But what about empiricity?

Putnam’s challenge was that logical empiricists had defined theoretical terms only in a negative way, as those which are not empirical. Of course, this alone is too weak. But it is an irreducible element. Much has been written on the problem raised by Putnam’s challenge, i.e., on the specific nature of theoretical terms. But there is, in my opinion, a much greater challenge for philosophy of science: what is empirical? – Indeed, our considerations show clearly that one cannot escape the problem of defining an adequate notion of empiricity. It is one of the fundamentals without which no philosophy of science which tries to give a reconstruction of empirical theories can stand. Indeed, to give an adequate explication of “empirical” is a ‘pragmatic’ problem – but this only means that it is a very complicated problem, nothing more. Of course, I cannot solve this problem here (a suggestion for a pragmatic definition of “observational term” can be found in my (1983), pp. 65–67).
But assume that both problems have been solved, and the structuralist criterion is strengthened by two requirements in the following way (for simplicity we use the B-criterion, but of course the same can be done with the G-criterion):

**DEFINITION 20.** A term $t$ of theory $T$ is $T$-theoretical* iff it is (1) $T$-theoretical$_B$ (i.e., $T$-internally measurable) and (2) not $T'$-theoretical$_B$ for some pretheory $T'$ of $T$, and (3) not empirical.

Note that "$T$-theoretical*" implies (but is not implied by) "$T$-theoretical according to Sneed" in the reconstruction of Balzer/Moulines (1980). Would this strengthened criterion solve the 'challenge of Putnam', namely to explicate the specific nature and function of theoretical terms in scientific theories?

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14. THE PROBLEM OF DISTINCTION BETWEEN CONTENTFUL AND CONTENTLESS THEORETICAL TERMS

The answer of our question is again: *no*. For we would expect from an adequate criterion of theoreticity, which gives an answer to Putnam's challenge, that it satisfies the following requirement:

(R3) An adequate criterion for theoreticity should be able to distinguish between contentful (empirically useful) terms of scientific theories and purely contentless (ad hoc and empirically useless) terms.

The notion of 'contentful' vs. 'contentless' terms can at first be made more precise by some examples. For instance, force is clearly a contentful term of CPM. But now think of a religious fanaticist, who wants to divide all forces in two components: a *divine* force-component and a *satanic* force-component (because in his opinion the world is the result of the interplay of God and Satan). Furthermore, he wants to give his religious views a 'physical basis' by defining for each force kind $f_i$: $f_i(p, t) = f_{i,d}(p, t) + f_{i,s}(p, t)$. Let us call the CPM with these additional functions of divine and satanic force components and the additional axiom $(f_{i,d} + f_{i,s} = f_i)$ the 'religious CPM' or 'RCPM'. Then it clearly follows: divine force and satanic force are RCPM-theoretical*; even with the strengthened formulation of theoreticity* in Definition 20 above. Because $f_{i,d}(p, t)$ as well as $f_{i,s}(p, t)$ are determined by the other functions: $f_{i,d}(p, t)$ by $f_i(p, t)$ and $f_{i,s}(p, t)$, and
Examples of this kind could be given by the thousands: simply introduce some new ad hoc terms such that they are determinable by each other and/or the old terms. Consider for instance Feynman’s famous example (1974, pp. 12-2): why do we attribute accelerations to forces, and not also – as Aristotle did – constant changes in position to some special force, call it “gorce”, which is proportional to the velocity. Feynman answers, the reason for this is that “gorce” has no empirical content – there is no possibility of an independent measuring of “gorces”. But of course, if we would construct a CPM with “gorces”, call it GoCPM, then gorces would come out as GoCPM-theoretical in the same way as forces, because gorces are determinable by velocity up to a constant factor. Of course the same problem also affects Gähde’s criterion, even if strengthened in the way of Definition 20; and furthermore, it also affects Sneed’s original criterion – according to all these criteria, “divine” and “satanic” forces or “gorces” would be theoretical. The deeper reason for this inability of the structuralist criterion to distinguish between contentful and contentless terms of a theory is the following: the criterion is formulated independently of what is empirical in the theory. It reflects only the possibility of measuring some term by some other terms. This includes the possibility of circularities with respect to empiricity: it may be, that some or even all theoretical terms are measurable by each other, but no one is measurable by means of empirical terms alone. The conclusion from this, I think, is the following: for every contentful theoretical term $t$ of $T$ there must be some specialization of $T$ which enables the measurement of $t$ by empirical terms alone. If this is not possible, the term is not contentful in the sense of having no function in the task of describing empirical reality. So again, the conclusion here is to reinstate the value of independently determining what is empirical in a theory. But to give a really satisfying criterion of theoreticity, more is needed. We must also be able to distinguish between empirically contentful, but redundant or superfluous terms and such empirically contentful terms which are not redundant. For instance, Feynman’s “gorce” is measurable by velocity and hence empirically contentful, but it is superfluous in the sense, that “gorce” has no additional information or function above “velocity” itself – which is reflected in the fact that there is no possibility of measuring “gorces” independently of velocity. Contentful terms could then be defined, in a first approach, as empirically

measurable and non-redundant terms. Of course, the elaboration of this idea would require another paper (on my opinion, the problem of redundancy is only solvable by closer investigation of the nature of language used in informal axioms – for some suggestions, see Schurz, 1983, p. 362–363). 55

15. FINAL CONCLUSION

We have imposed three requirements on an adequate notion of T-theoreticity: (R1) empirical terms should come out as non-T-theoretical, (R2) pre-T-theoretical terms should come out as non-T-theoretical, and (R3) only contentful (empirically useful) terms and not contentless (ad hoc and empirically useless) terms should be T-theoretical. An ample and representative set of examples has shown us that the B-criterion as well as the G-criterion neither satisfy (R1) nor (R2). In general, the idea of a theory-internal determination of what is theoretical or non-theoretical is misleading: it has the consequence that empirical or pre-T-theoretical terms come out as T-theoretical. So an independent notion of (i) “empirical” and of (ii) “pre-T-theoretical” is needed (They are also needed for Sneed’s original criterion in order to eliminate its vagueness.) Furthermore, neither the B-criterion nor the G-criterion, nor their strengthening by (i) and (ii), nor Sneed’s original criterion are able to distinguish between contentful theoretical terms and contentless terms – hence, none of the structuralist criteria satisfy (R3). Therefore, the search for an adequate criterion of theoreticity must go on.

APPENDIX: BASIC STRUCTURALISTIC CONCEPTS

We give now the detailed definitions for the structuralistic concepts informally explained in Section 2:

DEFINITION 21. $S$ is a typified class of $\langle k, l, m, \tau_1, \ldots, \tau_m \rangle - \text{structures}$ iff 1) $k, m \in \mathbb{N}$, $l \in \mathbb{N}_0$, 2) $\tau_1, \ldots, \tau_m$ are $(k + l)$-types and 3) for all $x$: $x \in S$ iff there exists $D_1, \ldots, D_k$ and $A_1, \ldots, A_l$ and $r_1, \ldots, r_m$ such that $D_i (1 \leq i \leq k)$ and $A_j (1 \leq j \leq l)$ are non-empty sets and $x = \langle D_1, \ldots, D_k; A_1, \ldots, A_l; r_1, \ldots, r_m \rangle$ and for all $i \leq m$: $r_i \subseteq \tau_i(D_1, \ldots, D_k, A_1, \ldots, A_l)$. – Thereby, the $(k + l)$-types $\tau_i$ and
the sets $\tau_i(X_1, \ldots, X_{k+i})$ (with $D_i = X_i$ ($1 \leq i \leq k$), $A_j = X_{k+j}$ ($1 \leq j \leq l$)) are inductively defined as follows: (a) for every $i$ ($1 \leq i \leq k + l$), $i$ is an $(k+l)$-type and $i(X_1, \ldots, X_{k+l}) := X_i$, and (b) if $\tau_1$ and $\tau_2$ are $(k+l)$-types, then $\text{Pow}(\tau_1)$ and $(\tau_1 \times \tau_2)$ are $(k+l)$-types and $\text{Pow}(\tau_1)(X_1, \ldots, X_{k+l}) := \text{Pow}(\tau_1(X_1, \ldots, X_{k+l}))$ and $(\tau_1 \times \tau_2)(X_1, \ldots, X_{k+l}) := \tau_1(X_1, \ldots, X_{k+l}) \times \tau_2(X_1, \ldots, X_{k+l})$.

(cf. Balzer, 1985a, p. 10f; 1985b, p. 130). For an informal explanation cf. Section 2. The $(k+l)$-types fix the set-theoretical types of the $r_i$ (the sets $\tau_i(D_1, \ldots, A_i)$ correspond to Bourbaki's "echelon construction schemes" ("Leitermengen"), cf. Balzer, 1985b, p. 130). $S$ contains all structures of the same type; and these structures can be regarded as structures of a many-sorted higher-order language (in which the special mathematical axioms and definitions for the mathematical sets $A_i$ are assumed or presupposed as given).

DEFINITION 22. $\Sigma$ is a species of $\langle k, l, \tau_1, \ldots, \tau_m \rangle$-structures -- having the form $\langle D_1, \ldots, D_k; A_1, \ldots, A_l; r_1, \ldots, r_m \rangle$ -- iff (a) $\Sigma$ is a nonempty subset of a typified class of $\langle k, l, m, \tau_1, \ldots, \tau_m \rangle$-structures and (b) $\Sigma$ is invariant under 'canonical transformations' in the following sense: We call $\text{Pow}(\tau_i)$ the type of relation $r_i \subseteq \tau_i(D_1, \ldots, A_2)$ ($1 \leq i \leq m$; hence, if $\tau$ is the type of $r$, then $r \in \tau(D_1, \ldots, A_i)$). Let $f = (f_1, \ldots, f_k)$ such that every $f_i$ ($1 \leq i \leq k$) is a bijective mapping $f_i: D_i \to D'_i$. For any relation $r$ of $(k+l)$-type $\tau$ -- here including base sets and their elements! -- the $f$-transport $r^f$ of $r$ is defined inductively as follows: (a) if $\tau = i$ with $1 \leq i \leq k$, then $r^f := f_i(r)$; (b) if $\tau = j$ with $k < j \leq k + l$, then $r^f := r$; (c) if $\tau = (\tau_1 \times \tau_2)$ and hence $r = \langle r_1, r_2 \rangle$ (with $r_i \in \tau_i(D_1, \ldots, A_i)$, $1 \leq i \leq 2$), then $r^f := \langle r_1^f, r_2^f \rangle$, and (d) if $\tau = \text{Pow}(\tau')$, then $r^f := \{ x | x \in r \}$. (This definition induces a bijective mapping $f': \tau(D_1, \ldots, D_k; A_1, \ldots, A_l) \to \tau(D'_1, \ldots, D'_k; A_1, \ldots, A_l)$, $f(r) = r^f$ for every $(k+l)$-type $\tau$.) Now, invariance under canonical transformations means that for all $\langle f_1, \ldots, f_k \rangle$ and $D'_i$ such that $\forall i \leq k (f_i: D_i \to D'_i$ is bijective) it holds:

$$\langle D_1, \ldots, D_k; A_1, \ldots, A_l; r_1, \ldots, r_m \rangle \in \Sigma$$

iff $$\langle D'_1, \ldots, D'_k; A_1, \ldots, A_l; r'_1, \ldots, r'_m \rangle \in \Sigma.$$
maps only the sets $D_i$ bijectively into some other sets, whereas it leaves the mathematical auxiliary sets $A_j$ unchanged. For an informal explanation cf. Section 2.

**DEFINITION 23.** (a) $M_p$ is a class of potential models (for some theory) iff there exists a typified class $S$ of $(k, l, m, \tau_1, \ldots, \tau_m)$-structures (with $k > 0$, $l \geq 0$, $m > 0$) and for every $i \leq m$ a species of structures $S_i \subseteq S$ such that $M_p := \cap_{i=1}^{m} S_i$ and every $S_i$ is a characterization of the $i$-th relation alone – where the latter condition means in set-theoretical terms: For all $j \leq m$, $x$ and $y$: if $x = \langle D_1, \ldots, D_k; A_1, \ldots, A_l; r_1, \ldots, r_m \rangle \in S_i, j \neq i$ and $y \subseteq \tau_j(D_1, \ldots, A_l)$, then $x_{-j}[y] \in S_i$. (b) $M$ is a class of models (for some theory $T$) iff it is a subset of $M_p(T)$ which is a species of structures and not a class of potential models (according to (a)).


The satisfaction of these complicated definitions can easily be seen by considering the nature of the axioms characterizing the structures: If the axioms are such that (i) the only restrictions on the sets $D_i$ following from the axioms are restrictions concerning their cardinality; (ii) all mathematical entities are put into the auxiliary base sets, (iii) each of the axioms characterizing $M_p$ contains – besides base sets, auxiliary base sets and mathematical functions (like $+, \cdot$, etc) – at most one relation, whereas the additional axioms characterizing $M$ are not reducible to a conjunction of such single-relation-axioms – then the set-theoretical definitions of $M_p$ and $M$ are satisfied. We will try to make this more precise in the following two lemmas:

**LEMMA 2.** Let $\Sigma$ be a set of $(k, l, m, \tau_1, \ldots, \tau_m)$-structures characterized by informal axioms such that – given axioms and definitions of set theory and mathematics – every axiom is equivalent with one of the following forms: (a) $r_i \in \tau(D_1, \ldots, A_l)$ or $r_i: \tau(D_1, \ldots, A_l) \rightarrow \tau'(D_1, \ldots, A_l) \ (1 \leq i \leq m; \ \tau, \tau' k + l$-types)), or (b) $k_1 < (\leq) k_2 (k_1, k_2$ are cardinals), or (c) $Qx_1, \ldots, Qx_n(A)$, where $Q \in \{\exists, \forall\}$ and every atomic formula occurring in the matrix $A$ is of one of the following forms: (ci) $x_i \in \tau(D_1, \ldots, A_l), \tau$ a $k + l$-type, or (cii) $x_i \in \tau_i (1 \leq j \leq m)$ or (ciii) $R(t_1, \ldots, t_p)$, where every $t_i (1 \leq i \leq p)$ is of the form $\tau_i(\alpha_i), \tau_i$ is a function which has as its range an auxiliary base set,
and $\alpha_i$ is a string of variables of the appropriate type; and $R$ is a logically or purely mathematically defined $p$-ary relation, i.e., the definiens of $R(t_1, \ldots, t_p)$ contains besides the $t_i$ only relations purely defined on the auxiliary base sets (e.g., $R(t_1, \ldots, t_p)$ may express the functional relationship $r_1(x_1) = f(r_2(x_2), \ldots, r_m(x_m))$ where $f$ is a mathematical function; or $R$ may be the defined predicate "is infinitely times differentiable", "is greater than", etc.). Then $\Sigma$ is a species of structures.

Proof (sketch). Let $f$ be a canonical transformation (as described in Definition 22). Assume that structure $X$ satisfies the $\Sigma$-axioms. We must show also that its transformation $X^f$ satisfies the corresponding $\Sigma^f$-axioms. If the axiom is of type (a), then $X^f$ satisfies it, because if $r_i \in \tau(D_1, \ldots, A_i)$, then $r_i^f \in \tau(D_1^f, \ldots, D_k^f; A_1, \ldots, A_i)$ (cf. Balzer, 1985b, Lemma 1, p. 131), furthermore if $r$ is a function then $r^f$ is a function (induction on type). If the axiom is of type (b), $X^f$ satisfies it, because the cardinalities $|D_i|$ are preserved under the bijective mappings of base sets. If the axiom is of type (c), we have for the atomic formulas of the matrix three cases. Case (ci): $x_i \in \tau(D_1, \ldots, A_i)$ iff $x_i^f \in \tau(D_1^f, \ldots, D_k^f; A_1, \ldots, A_i)$ (cf. again Balzer, 1985b, Lemma 1, p. 131). Case (cii): $x_i \in r_i (1 \leq i \leq n, 1 \leq j \leq m)$ iff $x_i^f \in r_i^f$ (proof by induction on the type). Case (ciii): Since the relations are functions into auxiliary base sets, and the latter remain unchanged under $f$, it holds for each application of $r_i$ to given string of variables $\alpha$ (of the appropriate type) $r_i(\alpha) = r_i^f(\alpha^f)$ (proof by induction on the type).

Furthermore, all other symbols occurring in the axioms are purely logical or mathematical and hence remain unchanged under $f$. Therefore, by substitution of identicals, $\beta$ iff $\beta^f$, where $\beta$ and $\beta^f$ are the respective atomic formulas in case (ciii). By substitution of equivalents we thus get in all three cases $\gamma$ iff $\gamma^f$, where $\gamma$ and $\gamma^f$ are the matrices of the axioms for $X$ and $X^f$, respectively, in case (c). By the quantification rule, we get $\delta$ iff $\delta^f$, where $\delta$, $\delta^f$ are the respective axioms of $X$ and $X^f$ in case (c). Hence $\Sigma$ is a species of structures. □

The restrictions on the axioms imposed by the lemma are such that most of the theories (if not all) reconstructed by structuralism satisfy them. Of course, a more detailed and more general version of this lemma would be desirable – which would require a closer determination of the language (e.g., a first order logic with the Zermelo–Fraenkel's set-theoretical axioms, in which informal set-theory can be embedded; or a type-theoretical logic).
LEMMA 3. Let $M_p$, $M$ be species of $(k, l, m, \tau_1, \ldots, \tau_m)$-structures such that each axiom characterizing $M_p$ contains at most one relation $r_i$ $(1 \leq i \leq m)$, whereas the additional axioms characterizing $M$ are not set-theoretically equivalent with a conjunction of such single-relation-axioms. Then $M_p$ is a set of potential models and $M$ a set of models for some theory $T$.

Proof. Let $S_i$ be the class of all structures of the $M_p$-type which satisfy those axioms for $M_p$ which contain no relation different from $r_i$. Let $x \in S_i$. Then for all $y$ with $y \subseteq \tau_i(D_1, \ldots, A_i)$ and $j \neq i$, $x_{-j}[y]$ also satisfies those axioms, since $x_{-j}[y]$ does not differ from $x$ with respect to the terms occurring in those axioms. Hence $x_{-j}[y] \in S_i$. So $S_i$ is a characterization of the $i$th term in the sense of Definition 23a. Obviously, $M_p$ is the intersection of the sets $S_i$ $(1 \leq i \leq m)$, so $M_p$ is a class of potential models (for some theory $T$). Since the additional axioms for $M$ are not equivalent to a conjunction of single-relation-axioms, it follows from the above that $M$ is not a class of potential models, hence (because of $M \subseteq M_p$) it is a class of models (for $T$). □

NOTES

1 Cf. Sneed (1971, p. 31); cf. also Balzer (1985a, p. 140 and 1986, p. 72). “To presuppose” means here “to imply logically” (Stegmüller, 1973, p. 51; cf. also Balzer, 1986, p. 75). I will not speak here about the so called circularity of theoretical terms – for a recent discussion of this problem see Conceptus 21(52), 1987, consisting of an article of Gadenne, a reply of Balzer and a comment on gadget and Balzer by the author.

2 The term “empiricity” will be used from now on as a counterpart of the term “theoreticity”: “empiricity” denotes the property of being determinable by observable means in at least an approximative way (the ‘approximative’-clause is necessary for real valued functions like position to be empirical).


5 In (D2) (p. 473) Balzer and Moulines refer to the class of all measurement methods (or determination methods) for a given term $t$ of $T$. Formally such a measurement method is described as a subset of $T$’s potential models or partial substructures of $T$’s potential models determining $t$ by other functions (pp. 471–479). But not just any such subset is a measurement method, because ad hoc measurement methods can be invented which do not correspond to accepted scientific laws or theories, and they would trivialize or undermine the criterion (pp. 485–489). Therefore the criterion is relativized to the class of all formulas which use the basic concepts of $T$ and are scientifically
accepted at a given time – the so-called set of existing expositions of \( T \) (pp. 487–489). But note that the term “existing exposition of \( T \)” is somewhat misleading since the set of existing expositions of \( T \) may of course contain also formulas (written material in scientific textbooks) which extensionally represent model sets which are not subsets of the models of \( T \), and hence represent model sets of theories \( T' \neq T \). All what is required is that these ‘existing expositions’ are extensionally sets of partial substructures of potential models of \( T \), i.e., they use some basic concepts of \( T \). Cf. also Stegmüller (1986, p. 172).

Another problem due to intertheoretic relations is discussed and solved in Balzer/Moulines (1980, pp. 481–485). A further problem not sufficiently discussed in Balzer/Moulines (1980) seems to me to be the fact, that a measurement method of a term \( t \) of \( T \) may eventually not be describable as a set of (partial substructures of) potential models of \( T \) because it contains also concepts not occurring in \( T \) – e.g., if force functions of CPM are measured by electrodynamic means.

Gäihde has convinced me that the B-criterion cannot simply be considered as an improved successor of the G-criterion, so that it is better to speak of two different criteria.


Balzer (1985a, p. 141): “Damit scheint die Entwicklung der Theoretizitätsfrage zu einem ersten dauerhaften Resultat geführt zu haben”.

Balzer (1985a, p. 139): “Das Beispiel der theoretischen Terme zeigt in eindrucksvoller Weise, daß es auch in der Wissenschaftstheorie Fortschritte gibt”.


One reason is, that Gäihdes notion of “invariance requirements associated with a theory” is abstracted from physical theories like classical particle mechanics and not precisely defined for arbitrary theories. Cf. Section 12.

Note that we do not construe force as a function with an index set as third argument set, as Sneed originally did, but list in Balzer’s style (1985a, p. 19) every force kind explicitly in the model, since this is more intuitive. Note furthermore, that whereas Balzer for simplicity omits the existential clause “there exists a \( P, T, s, m, f_1, \ldots, f_n \)”, we do include it. The reason for this is that the existential clause is strictly speaking
necessary: the terms $P$, $T$, $s$, $m$, $f_1, \ldots, f_n$ must neither be regarded as constants, because their interpretation may vary, nor as variables which are implicitly universally quantified upon, because this would be logically incorrect. Hence it seems to me better to include the existential clause (as usual in Sneed 1971 and Stegmüller 1973). On the other hand, the terms $R$ and $R^3$ have a fixed mathematical meaning and hence may be regarded as constants -- therefore it is not necessary to quantify over these terms. Finally note that concerning $T$, Balzer simply puts $T = R$, whereas we prefer the more general requirement (2).

Furthermore, in the following $R^+_0$ is the set of nonnegative real numbers, $\mathbb{N}$ is the set of positive integers (natural numbers) and $\mathbb{N}_0$ the set of nonnegative integers.

For the following, cf. Balzer (1985a), pp. 8–18.

The semi-colons in $(D_1, \ldots, r_m)$ are used for more vivid representation; logically they can be replaced by commas, since their information is contained in the parameters $k$ and $l$.

Note that the so called set $M_{pp}$ of partial potential models (containing only non-$T$-theoretical terms) has to be determined by application of the $B$- (or $G$-) criterion of $T$-theoreticity to $(M_p, M)$ -- hence it is not a primitive, but a defined element of the core.

To be more precise: Here and in the following, "function $r$ is measurable" shall mean the same as "the values of $r$ are measurable for any arguments". Concerning terms (see below), "a term $t$ is measurable" shall mean the same as "every realization of $t$ is measurable".

Note that in Definition 2, measurement methods are more generally defined as subsets of $M_p$, whereas the $M$-i-invariant measurement methods, relevant for $T$-theoreticity (Definition 4), must of course be subsets of $M$. Also note that in Definition 2, "$\forall x \in M_p$" (the version in Balzer, 1985a, p. 38) can be put equivalently for "$\forall x \in B$" (the version in Balzer, 1985b, p. 133, 1986, p. 84).

Proof ($\Rightarrow$): Let $x \in M_p$ and $x_i[r], x_i[r'] \in B$. Let $z := x_i[r]$. Obviously $z \in B$, $z_i[r], z_i[r'] \in B$, hence $r \Rightarrow r'$ ($\Rightarrow$ is trivial).

Furthermore, "$\forall x$" (Balzer, 1985a, p. 50) can also be put equivalently for "$\forall x \in B$" and "$\forall r', r' \in \tilde{r}'$" (Balzer, 1985b, p. 133) equivalently for "$\forall r, r'". This definition is identical with Balzer's (1985b, p. 134). In (1985a, p. 143) he gives a different version which is equivalent to ours, as seen from his D112 and T64 (for D112, note that $x_i = y_i$ is equivalent with $x = y_i[r]$). Furthermore, in D111 of 1985a Balzer defines $M$-i-invariance for all $B \subseteq M_p$ by the additional condition $B \subseteq M$. We prefer his version of (1985b, p. 134) because here the definition of $T$-theoreticity itself contains the important requirement $B \subseteq M$. Concerning $T$-theoreticity, all versions are equivalent.

Balzer (1985, p. 142) writes that this holds for first order predicate logic, but it obviously holds also for any logical language in which the law of substitution of identicals holds.

Definition 4 is identical with D5c of Balzer (1985b, p. 134) since his conditions (1) and (3) of D5c are the definitions of "admissible measurement method". (Note that an admissible measurement method is a species of structures -- Definition 2 -- and hence nonempty.) For the equivalence of Definition 4 with his D114 of (1985a, p. 143) cf. Note 20.
I think that no full theory of space, but only a 'definitorial theory of metrics' is necessary for measuring space; of course, the measurement and its result can be explained by such a deeper physical theory.

The intended applications of IGP, on the other hand, have to be restricted to ranges within which gases behave 'ideally'.

Since $k$ is not a set we must use $\{k\}$ as auxiliary base set; cf. also Balzer 1985a, p. 10.

On p. 149 of his 1985a Balzer argues, that $p$ and $v$ should be considered as empirical, but $\mathcal{A}$ as theoretical; but in phenomenological gas theory $p$, $v$ as well as $\mathcal{A}$ are empirical – there is agreement upon that in science; only the mole number $n$ (because it depends on mass) must be considered as theoretical. It seems that Balzer (1987) changed his option, and now regards $p$, $v$ and $\mathcal{A}$ as empirical.

Note that although $|Q| = 1$ we cannot simply put $Q = \{B\}$ and omit the quantifier $\forall y \in Q$ in (9) and replace $y$ by $B$, because then OBW would be strictly speaking not invariant under canonical transformations (see Appendix, Definition 22). Therefore also Balzer's Axiom (2) in D60 in (1985, p. 73), namely $Q = \{B\}$, is strictly speaking not correct because if $x \in M$ and $f : Q \rightarrow Q'$ is bijective, then this axiom need not hold for $x_{Q'}$ – e.g., $Q'$ could be $\{B'\}$ with $B' \neq B$. This is also the reason why in our other examples of theories with base sets $D$ with $|D| = 1$ we have to quantify over $D (\forall x \in D)$.

$x_{p,i}$ with axioms (1), (3), (5)-(8), (10) are the models of Balzer's reconstruction of Ohm's theory with battery (1985a, p. 73), and $x_{D,\mathcal{A},u,i}$ with axioms (1)–(D[\mathcal{P}]) (substitution of $P$ for $D$ in (1)), (4), (5)–(D[\mathcal{P}]), (9) and $\exists x, y \in P(\forall x \neq l(y))$ are models of Balzer's theory of measuring resistance by length comparison (1985, p. 73).

Our reconstruction is equivalent with Gähde's; but we used some simplifications: firstly, we introduce force not in Sneed's but in Balzer's style (cf. Note 13), and secondly, we simply wrote $\ldots \exists s_{0} \in \mathbb{R}^{3} \ldots (s_{0})$ in (4) and not, as does Gähde, $\ldots \exists x_{0} \in T \ldots (s(p, t_{0}))$; i.e., our $s_{0}$ is Gähde's $s(p, t_{0})$. From the solution of the differential equation (see the proof of Theorem 4) it can be derived in our reconstruction that $\exists x_{0} \in T(s(p, t_{0}) = s_{0})$.

The restriction that the Hooke force should be within the hookean range of the spring is formulated by Gähde as a constraint (cf. 1983, p. 60). Equivalently, we could also define the constraint $Q := \{x \in \text{Pow}(M(HCPM)) | \exists f_{m} \in \mathbb{R}^{+} \forall y \in x \forall p \in P^{*} \forall t \in T^{(1)} (|f(t(p, t)| \leq f_{m}))$, omit the second conjunct of (i) in the definition of $B$ – call this set $B'$ – and then require $B \in \text{Pow}(B') \cap Q$.

Note that $s(p, t)$ can equivalently given by a cosinus-function – in this case, the argument of $\arctan$ changes numerator and denominator.

A detailed outline of approximation methods for more-electron-systems (atoms and homonuclear molecules) can be found in Schurz (1980).

I.e. $\forall f \in L^{2}(\mathbb{R}, \mathbb{C})(|f \in \mathbb{I}, f \neq 0 \forall \alpha, \beta \cdot f(\alpha, \beta) \cdot d^{2} \sin \alpha \cdot d \alpha d \beta < \infty)$. Cf. Kutzelnigg (1975, p. 252).

The simplification is made that the mass of the electron is negligible in relation to the mass of the nucleus, such that the reduced mass can be equated with the electron's mass (Kutzelnigg 1975, p. 70).

Cf. Kutzelnigg (1975, p. 253, 255) and also Zoubek (1987), who gives a more
general reconstruction of time dependent quantum mechanics with unspecified operators. If an operator is hermitean, it has real eigenvalues. In Axiom (7), $(\partial/\partial \sigma), (\partial/\partial \sigma)$ and $(\partial/\partial \beta)$ are the operators for partial differentiation (cf. Kutzelnigg, 1975, pp. 69f).

37 I.e., $\int \int \int_0 \Psi^*(r, \alpha, \beta) \cdot \Psi(r, \alpha, \beta)r^2 dr \sin \alpha d\alpha d\beta = 1$; necessary for the probability interpretation.

38 Cf. Gerthsen (1971), p. 31-34.

39 Gähde considers only a two particle collision. His condition (5a) of his IC of p. 35 follows from our condition (3) by Newton's second law. His condition (5b), which states that the final velocities are the same for the two particles, follows from our condition (4). But Gähde's description does not contain our (4). On p. 36 Gähde argues that after $t_2$ the particles, although adhering, have a 'minimal distance'. But I think, in the idealized picture of point masses, our condition (4) -- usual in physics -- is adequate. Note that Gähde's embedding of IC into CPM does not contain these conditions (1983, p. 140) -- instead, he uses a specialization relation. In conformity with the view of theory-elements and theory-net, we present directly the specialized theory-element of PACPM.

40 $s^*(p, t)$ (in German "zweifache Stammfunktion von $(\Sigma_{k=1}^n f(p, t)/m(p))$") can be defined as $\int_0 \int_0 ((\Sigma_{k=1}^n f(p, t))/m(p)) \, dt \, dr$ for arbitrary $t_1$ (the possible variation of $t_1$ is already contained in the possible variation of $k_1$ and $k_2$).


42 Of course, in relativistic mechanics the thing is a little bit different: here we have an absolute upper limit on velocity: the velocity of light.

43 Cf. Feynman (1973), p. 52-1f: translations in time and in space, rotations in space, and adding a constant velocity have all the same physical nature: they are physical symmetry operations. If we apply these symmetry operations to the process of measurement, we get the corresponding scale-invariances.

44 Invariance $=_{1}$ of term $r$ is finer than $=_{2}$ iff for all $r \in R$, $\{r' \in R | r' =_{1} r\} \subseteq \{r' \in R | r' =_{2} r\}$.

45 "Man muß also noch ein Bezugsystem auszeichnen, um $s(p, t)$ wirklich bestimmen zu können. Die Auszeichnung kann aber nicht in dem formal durch $M_p(KPM)$ abgesteckten Rahmen erfolgen, da dort nicht von verschiedenen Bezugsystemen geredet wird. Eine solche "Meßmethode", die sich nicht durch Fixierung bestimmter, im Rahmen der Theorie zur Verfügung stehender Parameter wirklich eindeutig machen läßt, wollen wir nicht als Meßmethode bezeichnen" (1985a, p. 51).

46 The Galilei-transformations have to be formulated now with 'minus' instead of 'plus' since they are applied to coordinate systems, not bodies.

47 The equivalence of our Definition 15 to Gähde's definitions on pp. 131-135 is made obvious by the following hints: Gähdes "Funktionsvariable $\phi$" in "Schritt 6" (p. 135) corresponds to our "term $r$". His condition "es gibt eine Ergänzungsfunktion $e^* \in ZEF^+$, sodaß $\phi$ ergänzende Funktionsvariable bzgl. $e^*$ ist" is equivalent to our
requirement that a subset \( \mu \subseteq \{1, \ldots, m\} \) with \( i \in \mu \) exists – which is clear from his definitions in “Schritt 1” and “Schritt 2” (p. 131–133). His additional requirement in “Schritt 5”, namely “\( AE^T(e^*) \)”, is defined by Conditions (2) and (3) of the definition in his “Schritt 5” (p. 135). From his “Schritt 1” (p. 131f) it follows that his “\( D_1(e^*) \)” in “Schritt 5” is exactly our \( M_{p_r, \mu} \) of our Definition 15. From “Schritt 4” (p. 134f) it can be seen that his \( \left| e^*(z) \cap M \| Z_T > 1 \right| \) is equivalent to the part of our Condition (1) behind “\( \forall x \in M_{p_r, \mu} \)”.

Hence his condition (2) of the definition in “Schritt 5” is equivalent to our Condition (1) of Definition 15. In “Schritt 3” Gähde defines the proposition “\( ZPVT(M) \)” which is equivalent to our (2) and (2a). Furthermore note that our (2b) is equivalent to “\( \exists x \in M_{p_r, \mu} \exists y \in B \forall z \in B((x = y_{-\mu}) \land ((y_{-\mu} = z_{-\mu}) \rightarrow \rho_\mu \equiv \rho_\mu')) \)”.

From this and the same consideration as above it is seen that Gähde’s Condition (3) of the definition in “Schritt 5” is equivalent to Condition (2b) of our Definition 15.

48 Let us briefly mention the modified version of Gähde’s criterion presented in Stegmüller (1986, pp. 173–177). It differs from Gähde’s original criterion (in Definition 15) by the following modifications: (i) From Condition (1) of Definition 15 it follows that every element of \( M_{p_r, \mu} \) must have an extension to a model of \( M \). Since intuitively the relations in \( r_\mu \) are \( T \)-theoretical relations, whereas the other relations are the empirical (non-\( T \)-theoretical) ones, this means that the core theory element \( T \) has no empirical content. Indeed Gähde proves this for CPM (1983, p. 98), but as a general requirement this seems not adequate. Stegmüller (1986, p. 176; D6–9, (3)) weakens this requirement to “\( \forall x \in M_{p_r, \mu} (3z \in M(x_{-\mu} = x) \rightarrow \exists y \in M(\ldots)) \)” where “\( (\ldots) \)” stands for the matrix of (1). Note that this modification of (1) is equivalent with “\( \forall x \in M \exists y \in M(x_{-\mu} = y_{-\mu} \land \lnot (r_\mu \equiv r_\mu')) \)” which is the negation of (2b), if “\( M \)” is substituted for “\( B \)”.

(ii) Instead of (2b), Stegmüller (p. 176, D6–9, (b) and p. 175, D6–7) uses Balzer’s stronger definition of measurement method, i.e., the quantifier part is “\( \forall x, y \in B \)” instead of “\( \exists x \in B \forall y \in B \)”; furthermore (iii) \( B \) is required to be a species of structures (p. 175, D6–7, (2)); and finally (iv) in \( B \) the relations in \( r_\mu \) should be ‘genuinely dependent’ on the other relations, i.e., it should not hold that “\( \forall x, y \in B (r_\mu = r_\mu') \)” (p. 175, D6–7, (4)). Independent from the question of whether all these modifications are improvements we want to point out that our following arguments hold also for this modified version.

49 For instance: Let \( a(d, t) := k \cdot n(d, t) \cdot s(d, t); p(d, t) := 2 \cdot a(d, t); v(d, t) := a(d, t)/2 \) (for all \( t \in T \) and the only \( d \in D \)). If there exists a \( t_1 \in T, t_2 \in T \) with \( t_1 \neq t_2 \) such that \( a(d, t_1) > 0 \) and \( a(d, t_2) > 0 \) (normal case), then let \( p^*(d, t) := \exp(t) \cdot p(d, t) \) and \( v^*(d, t) := v(d, t)/\exp(t) \). If not (abnormal case), then take some arbitrary \( t_1 \in T, t_2 \in T \) with \( t_1 \neq t_2 \) and \( a(d, t_1) = a(d, t_2) = 0 \) and let \( p^*(d, t_1) := 0 \) and \( v^*(d, t_1) := 0 \); otherwise \( p^* = p \) and \( v^* = v \). In both cases it holds: (a) \( \exists r \in R \forall d \in D \forall t \in T (p^*(d, t) = r \cdot p(d, t)) \), (b) \( \exists r \in R \forall d \in D \forall t \in T (v^*(d, t) = r \cdot v(d, t)) \), (c) \( \forall d \in D \forall t \in T (p(d, t) \cdot v(d, t) = a(d, t)) \), and (d) \( \forall d \in D \forall t \in T (p(d, t) \cdot v(d, t) = a(d, t)) \).
Under this assumption, that the position function can be theoretical $\Theta$ is of course excluded apriori, because if $s$ would be determined in some $B$ up to multiplication with a positive constant or linear transformation, then $B$ would not be admissible with respect to $\sim$ since multiplication with a positive constant and linear transformation are finer invariances (cf. Note 44) than Galilean invariance.

By the same method as in Note 49 it can be shown that for every $x \in M(CPM^n)_{p,-}$ there exist mass functions $m$, $m'$ and acceleration functions $a$, $a'$ such that $m$ and $m'$ as well as $a$ and $a'$, respectively, are not equivalent with respect to invariance of multiplication with a positive constant factor, and it holds for all $p \in P^x$, $i \in T^x \sum_{i=1} f_i(p, t) = m(p) \cdot a(p, t) = m'(p) \cdot a'(p, t)$. By integrating $a$ and $a'$ twice after the time it follows that $s$ and $s'$ are not equivalent with respect to Galilean invariance.

I.e., the oscillation system may now move with constant velocity $v_0$. The Galilei-transformation of $s(p, t)$ yields $s'(p, t) = k_1 t - k_2$ (for $k_1, k_2 \in \mathbb{R}^3$) which again satisfies (4).

Note that only a few Boolean algebras can simultaneously satisfy Axiom (7) of Definition 14 and $\forall x \in B(U(x) = P(x))$.

The fruitful applicability of the notion of "nonredundant relevant theory-consequence" to the problem of versimilitude is shown in Schurz/Weingartner (1987).

In the following $D_1, \ldots, A_i$ stands as an abbreviation for $D_1, \ldots, D_k$; $A_1, \ldots, A_i$.

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